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A Comparison on the MRL Performances of Optimal MEWMA and Optimal MCUSUM Control Charts

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***Abstract*— The MEWMA (called the multivariate exponentially weighted moving average) chart and the MCUSUM (called the multivariate cumulative sum) chart are used in process monitoring when a quick detection of small or moderate shifts in the mean vector is desired. The primary objective of this study is to compare the performances of the optimal MEWMA and optimal MCUSUM charts based on their median run length (MRL) profiles. The number of quality characteristics considered is p = 2. Two cases are studied, i.e., Case 1 (a shift in only one variable) and Case 2 (a shift in two variables). A Monte Carlo simulation is conducted using Statistical Analysis Software (SAS) to study and compare the MRL performances for various magnitudes of mean shifts when the process is normally distributed. Overall, the results show that the MRL performances of the MEWMA and MCUSUM charts are comparable.**

***Keywords*— Median run length (MRL), MEWMA, MCUSUM**

I. INTRODUCTION

Many semiconductor manufacturers and chemical process plants maintain manufacturing databases on hundreds of variables. Often the total size of these databases is measured in millions of individual records. The monitoring of these data with univariate statistical control charts is often misleading. The use of multivariate methods has increased greatly in recent years for this reason, as mentioned in [1].

Hotelling [2] first introduced the Hotelling's T^2 multivariate quality control chart for monitoring the process mean vector. Since then, the application of the Hotelling's T^2 control chart as a process monitoring tool has became increasingly popular in the field of statistical process control (SPC). It is common to monitor several related variables or quality characteristics simultaneously with the introduction of modern data-acquisition equipments and computers (e.g. [3]).

The Hotelling's T^2 control chart which can be used in both Phase I and Phase II situations is a Shewhart-type control chart. It uses information only from the current sample and ignores any information given by the entire sequence of samples. As a result, it is insensitive to small and moderate shifts in the mean vector. Two other types of multivariate control charts which is based on a Phase II procedure, i.e., the multivariate exponentially weighted moving average (MEWMA) and multivariate cumulative sum (MCUSUM) charts, are proposed as superior alternatives to the Hotelling's T^2 chart, where they take into account of the present and past information about the process.

Optimal statistical designs of the MEWMA and MCUSUM charts, based on average run length (ARL) and median run length (MRL) have been proposed, as in [4] and [5]. The MRL, which is the 50th percentage point of the run length distribution is suggested to be used as a potential alternative to the ARL. This is due to the fact that the in-control run length distributions of the MEWMA and MCUSUM charts are highly skewed, hence interpretation based on the ARL can be misleading. In addition, the skewness of the run length distribution changes according to the magnitude of the shift in the mean vector and this makes interpretation based on ARL more complex. Thus, MRL is used to evaluate the performances of control charts in this study.

Since both the MEWMA and MCUSUM charts have comparable ARL performances, as in [6], we are interested to compare the performances of the optimal MEWMA and optimal MCUSUM charts, based on their MRL profiles, for monitoring of the mean vector of a multivariate normally distributed process. An efficient control chart is able to detect instantaneously a process shift and adopts the essential corrective actions to improve the process quality. The MRLs are obtained using a Monte Carlo simulation. The mean vector is allowed to change for various sizes of shifts, so that the performances of the charts, based on MRL can be compared.

The remainder of this paper is organized as follows: Sections II, III, IV and V review the MEWMA chart, optimal design of the MEWMA chart, MCUSUM chart and optimal design of the MCUSUM chart, respectively. A simulation study is carried out to compare the MRL performances of the optimal MEWMA and MCUSUM charts in Section VI. Finally, some useful conclusions are summarized in Section VII.

II. The MEWMA Control Chart

The MEWMA statistics proposed in [7] is defined as follows:

$$\boldsymbol{Z}_t = r\boldsymbol{X}_t + (1-r)\boldsymbol{Z}_{t-1}, \; t = 1, 2, \ldots, \tag{1}$$

where $\boldsymbol{Z}_0 = \boldsymbol{0}$ is the initial vector of the MEWMA statistic and r ($0 < r \leq 1$) is the smoothing constant. Let $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots,$ be the independently and identically distributed multivariate normal random vectors, each with p components. It is assumed without loss of generality that the on-target process mean vector, $\boldsymbol{\mu}_0$ is a vector of zeros.

The plotted values on the MEWMA chart are

$$T_t^2 = \boldsymbol{Z}_t' \, \Sigma_{Z_t}^{-1} \, \boldsymbol{Z}_t, \; t = 1, 2, \ldots, \tag{2}$$

where $\Sigma_{Z_t} = \frac{r}{2-r}\left[1-(1-r)^{2t}\right]\Sigma_{X_t}$ is the covariance matrix of $\boldsymbol{Z}_t$ and Σ_{X_t} is the covariance matrix of $\boldsymbol{X}_t$. The MEWMA statistic reduces to the Hotelling's T^2 statistic if $r = 1$.

The MEWMA control chart gives an out-of-control signal as soon as

$$T_t^2 > H, \quad t = 1, 2, \ldots, \tag{3}$$

where the control limit $H > 0$ is a constant that is chosen to achieve the desired in-control MRL (MRL_0).

III. The Optimal Design of a MEWMA Chart

A MEWMA chart is optimal in detecting a shift if it has the smallest out-of-control MRL (MRL_1) among all MEWMA charts with the same MRL_0 for the same magnitude of shift.

The optimal design of a MEWMA chart specifies the optimal selection of the constants, r and its corresponding control limit, H. A table providing the optimal combinations of r and H for selected number of variables, in-control ARLs and sizes of shifts, for the optimal MEWMA chart is given in [8]. Since the optimal combinations of r and H given in [8] are limited, these optimal combinations are extracted from the graphs given in [4]. Due to space constraint, the graphs in [4] are not shown in this paper. Interested readers can request this paper from its second author.

IV. The MCUSUM Control Chart

Several researchers have developed multivariate extensions of the univariate CUSUM chart (e.g. [9], [10], [11], [12], [13], [14]). Two MCUSUM charts are proposed in [9], where the one with the better ARL performance is based on the following statistics:

$$C_t = \sqrt{\left(\boldsymbol{S}_{t-1} + \boldsymbol{X}_t - \boldsymbol{a}\right)' \Sigma_{X_t}^{-1} \left(\boldsymbol{S}_{t-1} + \boldsymbol{X}_t - \boldsymbol{a}\right)}, \; t = 1, 2, \ldots, \tag{4}$$

and

$$\boldsymbol{S}_t = \begin{cases} \boldsymbol{0}, & C_t \leq k \\ \left(\boldsymbol{S}_{t-1} + \boldsymbol{X}_t - \boldsymbol{a}\right)\left(\boldsymbol{1} - \dfrac{k}{C_t}\right), & C_t > k \end{cases} \tag{5}$$

where $\boldsymbol{a}$ is the aim point or target value for the mean. In this paper, it is assumed without loss of generality that $\boldsymbol{a} = \boldsymbol{0}$. Note that $\boldsymbol{S}_0 = \boldsymbol{0}$ and $k > 0$ is the reference value of the scheme. According to [9], an out-of-control signal is generated when

$$Y_t = \sqrt{\boldsymbol{S}_t' \, \Sigma_{X_t}^{-1} \, \boldsymbol{S}_t} > H', \tag{6}$$

where $H' > 0$ is the control limit.

The MCUSUM procedure is often based on the assumption that the observation $\boldsymbol{X}_t$, for $t = 1, 2, \ldots, p$, belongs to an independently and identically distributed process, from a multivariate normal distribution.

V. The Optimal Design of a MCUSUM Chart

An optimal MCUSUM chart is defined as the chart that has the smallest out-of-control MRL (MRL_1) at a particular shift in the process mean, among all the MCUSUM charts with the same MRL_0.

The optimal parameter k of the MCUSUM chart, determined based on ARL as a criterion to be minimized, is approximately given as half of the size of the shift, λ, where λ is the noncentrality parameter given as follows [9]:

$$\lambda = \sqrt{(\boldsymbol{\mu} - \boldsymbol{\mu}_0)' \, \Sigma_{X_t}^{-1} (\boldsymbol{\mu} - \boldsymbol{\mu}_0)}, \tag{7}$$

This value of k appears to minimize the ARL_1 for a particular magnitude of shift, based on a given ARL_0. After obtaining the optimal reference value of k, one needs to obtain the corresponding control limit, H'. The optimal parameters, k and H' for an optimal MCUSUM chart, based on the desired magnitude of shift for a quick detection and MRL_0, are taken from [5]. Note that the tables giving the optimal parameters are not displayed here because of space limitation. However, this paper containing the tables can be requested from its second author.

VI. Comparison of MRL PERFORMANCES: MEWMA Versus MCUSUM Charts

The main aim of SPC techniques is to detect as early as possible the presence of assignable causes of variation that affect the quality of a process. This study compares the optimal MEWMA and optimal MCUSUM charts for monitoring the mean vector by assessing how their MRLs differ. When the process is in-control, the MRL should be sufficiently large to minimize false alarms. When the process is out-of-control, the chart should have a small MRL, so that process shifts can be detected promptly. The performances of the MEWMA and MCUSUM charts are compared by setting $\text{MRL}_0 = 370$ for both charts and compare their MRL_1s, for a given shift in the process. The magnitudes of shifts in the mean vector are fixed at $\lambda \in \{0, 0.05, 0.1, 0.15, 0.2, 0.25, 0.3, 0.35, 0.4, 0.45, 0.5, 0.75, 1, 2, 4, 6, 8, 10, 12, 18\}$. The number of quality characteristics considered is p=2.

Two cases are considered in this paper, where Case 1 involves a shift of only one variable while Case 2 involves a shift of two variables. The out-of-control mean vectors for Cases 1 and 2 are respectively, $\boldsymbol{\mu}_s = (\delta , 0)'$ and $\boldsymbol{\mu}_s = (\delta , \delta)'$.

The values of MRL_1 for the MEWMA and MCUSUM charts are computed via simulation using the SAS program. Here, MRL_1 is defined as the median number of bivariate observations that must be plotted on the MEWMA (or MCUSUM) chart before the chart signals an out-of-control, when a process shift occurs. To explain how MRL_1 values are computed, the MEWMA chart is considered and explained as follows: First, a sequence of bivariate normal observations are generated for Cases 1 and 2, followed by computing the MEWMA statistics using the formulae in Equations (1) and (2). When the MEWMA chart issues an out-of-control ($T_t^2 > H$, for t = 1, 2, …), say at $t = t_0$, the value of t_0 is recorded as the run length value corresponding to the trial. This process is repeated for 5000 trials, where a run length value is computed for each trial. Then the median of the 5000 run length values, obtained from the 5000 trials is taken as the MRL_1 value. The same procedure is used to obtain the MRL_1 values for the MCUSUM chart.

Table I shows the MRL_1s for the MEWMA and MCUSUM charts. For the MEWMA chart with MRL_0 = 370 and a shift of $\lambda = 0.1$, for which a quick detection is desired, we obtain r = 0.008 and H = 5.6 from the graphs given in [4]. Similarly, for the MCUSUM chart with MRL_0 = 370 and a shift of $\lambda = 0.1$, for which a quick detection is needed, we obtain k = 0.09 and H = 18.52 from the table given in [5]. The optimal parameters in Tables II to IV are obtained using the same method. Tables I, II, III and IV give MRL_1 results for optimal λ's of 0.1, 0.3, 0.5 and 1, respectively.

The MEWMA and MCUSUM charts are more effective in detecting small shifts. For example, consider the MCUSUM chart in Case 1 of Table I. When λ increases from 0.05 to 0.1, the MRL_1 reduces significantly from 309 to 201. The MRL_1 decreases further from 201 to 140 when λ increases from 0.1 to 0.15. Then when λ increases from 2 to 10, the chart's MRL_1 decreases from 10 to 2, i.e., at a slower pace compared to the speed of decrease in MRL_1 discussed above. A similar trend is also observed for the MEWMA chart. Thus, the effectiveness of the MEWMA and MCUSUM charts in the detection of shifts are more pronounced when λ is small.

Next, we compare the performances of the optimal MEWMA and optimal MCUSUM charts in Table I. In Case 1, the rate of a decrease in the MRL_1s for the MEWMA and MCUSUM charts are almost the same regardless of the value of λ. In other words, the MRL_1 for the MEWMA chart is approximately the same as that of the MCUSUM chart when λ is a constant. For example, when λ = 0.2, the MRL_1 for Case 1 of the MEWMA chart is 104 while that of the MCUSUM chart is 105. When λ = 0.5 onwards, the MRL_1s of both charts for Case 1 are exactly the same. Similarly, in Case 2, although differences between the MRL_1s of the MEWMA and MCUSUM charts exist, the differences are small and insignificant.

The MRL_1 trends of the two charts in Table I, mentioned above are also observed in Tables II to IV. Therefore, we conclude that the optimal MEWMA and optimal MCUSUM control charts have similar performance in detecting out-of-control signals.

It is worth noting that since the MRL_1 values for Cases 1 and 2 of the MEWMA and MCUSUM charts are about the same, for the same value of λ, a graphical illustration of MRL_1 versus λ, based on the results in Tables 1 – 4 is less meaningful as the plots for the MEWMA and MCUSUM charts will overlap. There is negligible difference in the MRL_1s of the two charts when λ has the same value.

VII. CONCLUSIONS

Based on the MRL comparison, we conclude that both the optimal MEWMA and optimal MCUSUM charts are equally efficient in detecting shifts. The MRL performances of the optimal MEWMA and optimal MCUSUM charts are comparable in detecting out-of-control signals for Case I (a shift in one variable) and Case 2 (a shift in two variables).

The MRL can be used as an alternative or as a secondary criterion to the ARL, in the evaluation of the performances of MEWMA and MCUSUM charts.

In this study, we only consider the use of MRL to evaluate the performances of optimal MEWMA and optimal MCUSUM charts, which involve two quality characteristics (p = 2). Further research can be made by increasing the number of quality characteristics, p in studying the MRL performances of the two charts. It is also useful to study the existing methods and suggest superior ones, in determining which of the p different variables contribute to an out-of-control signal, when one occurs.

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TABLE I

MRL$_1$S OF THE OPTIMAL MEWMA AND OPTIMAL MCUSUM CHARTS WITH MRL$_0$ = 370, p = 2, ρ = 0 AND A SHIFT, λ = 0.1

Case 1 (a shift in one variable)			
λ	**δ**	**MEWMA (r = 0.008, H = 5.6)**	**MCUSUM (k = 0.09, H = 18.52)**
0.00	0.00	371	370
0.05	0.05	301	309
0.10	0.10	199	201
0.15	0.15	137	140
0.20	0.20	104	105
0.25	0.25	83	84
0.30	0.30	68	69.5
0.35	0.35	59	59
0.40	0.40	52	52
0.45	0.45	45	46
0.50	0.50	41	41
0.75	0.75	27	27
1.00	1.00	20	20
2.00	2.00	10	10
4.00	4.00	5	5
6.00	6.00	4	4
8.00	8.00	3	3
10.00	10.00	2	2
12.00	12.00	2	2
18.00	18.00	2	2
Case 2 (a shift in two variables)			
λ	**δ**	**MEWMA (r = 0.008, H = 5.6)**	**MCUSUM (k = 0.09, H = 18.52)**
0.00	0.00	371	370
0.05	0.04	300	303.5
0.10	0.07	200	205
0.15	0.11	139	141
0.20	0.14	103	105
0.25	0.18	83	84
0.30	0.21	69	70
0.35	0.25	58	59
0.40	0.28	51	51
0.45	0.32	45	46
0.50	0.35	41	41
0.75	0.53	27	27
1.00	0.71	20	20
2.00	1.41	10	10
4.00	2.83	5	5
6.00	4.24	4	4
8.00	5.66	3	3
10.00	7.07	2	2
12.00	8.49	2	2
18.00	12.73	2	2

TABLE II

MRL$_1$S OF THE OPTIMAL MEWMA AND OPTIMAL MCUSUM CHARTS WITH MRL$_0$ = 370, p = 2, ρ = 0 AND A SHIFT, λ = 0.3

Case 1 (a shift in one variable)			
λ	**δ**	**MEWMA (r = 0.015, H = 7.1)**	**MCUSUM (k = 0.225, H = 11.64)**
0.00	0.00	369	370
0.05	0.05	312	309
0.10	0.10	198	222
0.15	0.15	135	148
0.20	0.20	99	103
0.25	0.25	76	76
0.30	0.30	63	61
0.35	0.35	54	50
0.40	0.40	46	41
0.45	0.45	40	36
0.50	0.50	36	32
0.75	0.75	23	20
1.00	1.00	17	14
2.00	2.00	9	7
4.00	4.00	4	4
6.00	6.00	3	2
8.00	8.00	2	2
10.00	10.00	2	2
12.00	12.00	2	1
18.00	18.00	1	1
Case 2 (a shift in two variables)			
λ	**δ**	**MEWMA (r = 0.015, H = 7.1)**	**MCUSUM (k = 0.225, H = 11.64)**
0.00	0.00	369	370
0.05	0.04	306	305
0.10	0.07	206.5	222
0.15	0.11	137	149.5
0.20	0.14	99	102
0.25	0.18	77	77
0.30	0.21	63	60
0.35	0.25	53	49
0.40	0.28	46	42
0.45	0.32	40	36
0.50	0.35	36	32
0.75	0.53	23	20
1.00	0.71	17	14
2.00	1.41	8	7
4.00	2.83	4	4
6.00	4.24	3	2
8.00	5.66	2	2
10.00	7.07	2	2
12.00	8.49	2	1
18.00	12.73	1	1

TABLE III
MRL$_1$S OF THE OPTIMAL MEWMA AND OPTIMAL MCUSUM CHARTS WITH MRL$_0$ = 370, p = 2, ρ = 0 AND A SHIFT, λ = 0.5

Case 1 (a shift in one variable)			
λ	**δ**	**MEWMA (r = 0.06, H = 10.06)**	**MCUSUM (k = 0.35, H = 8.68)**
0.00	0.00	370	371
0.05	0.05	324	321
0.10	0.10	245	252
0.15	0.15	175	175
0.20	0.20	121	121
0.25	0.25	88	87
0.30	0.30	65	66
0.35	0.35	52	52
0.40	0.40	42	41
0.45	0.45	35	35
0.50	0.50	30	30
0.75	0.75	18	17
1.00	1.00	12	12
2.00	2.00	6	6
4.00	4.00	3	3
6.00	6.00	2	2
8.00	8.00	2	2
10.00	10.00	1	1
12.00	12.00	1	1
18.00	18.00	1	1
Case 2 (a shift in two variables)			
λ	**δ**	**MEWMA (r = 0.06, H = 10.06)**	**MCUSUM (k = 0.35, H = 8.68)**
0.00	0.00	370	371
0.05	0.04	330	323
0.10	0.07	247	250
0.15	0.11	175	175
0.20	0.14	123	125
0.25	0.18	87	86
0.30	0.21	65	65
0.35	0.25	50	50
0.40	0.28	42	41
0.45	0.32	35	34
0.50	0.35	30	30
0.75	0.53	17	17
1.00	0.71	12	12
2.00	1.41	6	6
4.00	2.83	3	3
6.00	4.24	2	2
8.00	5.66	2	2
10.00	7.07	1	1
12.00	8.49	1	1
18.00	12.73	1	1

TABLE IV
MRL$_1$S OF THE OPTIMAL MEWMA AND OPTIMAL MCUSUM CHARTS WITH MRL$_0$ = 370, p = 2, ρ = 0 AND A SHIFT, λ = 1.0

Case 1 (a shift in one variable)			
λ	**δ**	**MEWMA (r = 0.16, H = 11.53)**	**MCUSUM (k = 0.675, H = 5.16)**
0.00	0.00	370	370
0.05	0.05	342	353.5
0.10	0.10	287	294.5
0.15	0.15	218	237
0.20	0.20	165	181
0.25	0.25	128	139
0.30	0.30	99	107
0.35	0.35	76	80
0.40	0.40	60	62
0.45	0.45	47	48
0.50	0.50	38	39
0.75	0.75	17	17
1.00	1.00	11	10
2.00	2.00	4	4
4.00	4.00	2	2
6.00	6.00	2	1
8.00	8.00	1	1
10.00	10.00	1	1
12.00	12.00	1	1
18.00	18.00	1	1
Case 2 (a shift in two variables)			
λ	**δ**	**MEWMA (r=.16, H=11.53)**	**MCUSUM (k=.675, H=5.16)**
0.00	0.00	370	370
0.05	0.04	342	351
0.10	0.07	287	293
0.15	0.11	224	239.5
0.20	0.14	175	185
0.25	0.18	130.5	142
0.30	0.21	97	107
0.35	0.25	74	81
0.40	0.28	57	61
0.45	0.32	47	48
0.50	0.35	37	38
0.75	0.53	17	17
1.00	0.71	11	10
2.00	1.41	4	4
4.00	2.83	2	2
6.00	4.24	2	1
8.00	5.66	1	1
10.00	7.07	1	1
12.00	8.49	1	1
18.00	12.73	1	1

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