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A Study on the Estimation of the Frobenius Numbers Generated by Binomial Coefficients Using Linear Regression

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***Abstract*—The Frobenius problem is a classical problem in number theory and combinatorics that explores the range and maximum values of integers that can be represented as combinations of a given set of integers. There is a simple formula for the Frobenius Number for the case of two integers. There are just some special results for the cases of three or more integers, and the general formula has not been discovered. In this paper, we study the approximation of Asymptotic behavior using Linear Regression to get a Frobenius Number for one existing and a new result. Initially, finding the Frobenius number required a lot of computation, including finding an Apery set. Still, we took advantage of the fact that the Frobenius number can be found directly by making a function prediction using the individual data of the found Frobenius numbers. The main reason why function prediction in this way can be correct is that the calculation to find the Frobenius number involves a non-negative integer combination of the elements of the numerical semigroup, so if we think of it as a non-negative integer combination with the coefficients of the integers closest to the function found in Linear Regression, we can get a predicted function that is expected to be accurate. The methodology of this study may not be well applied to functions with general real numbers. Still, we found that if we analyze discrete values well, we can get a sufficiently predicted function.**

***Keywords*—Frobenius number; apery set; asymptotic behavior; linear regression.**

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I. INTRODUCTION

The Frobenius problem is a subject that has been studied continuously in number theory for a very long time until the present day. This problem is also related to the expression of various structures, in that the analysis of a given numerical sample group is performed. When there is a set of disjoint natural numbers, consider numbers that can be written as their nonnegative integral combination. Numbers above a certain number can always be expressed at this time. For example, thinking about 3 and 4, we see that more than six natural numbers can always be expressed. Or thinking about 4, 5, 6, and 8, and more numbers can always be expressed.

Given two mutually prime natural numbers, it is known that the Frobenius number is easy to find. However, there is no general solution for finding the Frobenius number when given three or more mutually prime natural numbers, and the Frobenius number of a numerical semigroup of three or more mutually prime natural numbers has only been limited to the exceptional cases.

Exact solutions are not generally available in a closed-form expression for three coprime positive integers. However, significant progress has been made through computational methods. For instance, the algorithmic approach for solving the Frobenius number problem for three or more integers was notably advanced in the early 2000s, when efficient algorithms utilizing integer programming techniques were developed [1]. Some recent works are on the Frobenius number of numerical semigroups for special cases. In [2], the authors focus on the Frobenius number generated by the following numerical semigroup.

$S_c = \{a, ha + d, ha + c + bd, ha + 2c + b^2 d, \dots, ha + kc + b^k d\}$. In addition, there are a number of recent related studies that deal with a set commonly created by a nonnative integrator combination of three or more disjoint numbers. The statement is not complex, but it is by no means easy from the standpoint of problem solving, because the problems have to be dealt with not only in finding the Frozenius number itself but also various other indicators. In common with the research results, a standardized technique was used after obtaining the Apery set. However, it can be seen that the pre-processing before obtaining the set is still not standardized and very diverse [4]-[10].

In addition, as an application method of the Provenius problem, research results deal with Covariate, distribution of generus, and multiplicity using it [11]-[13]. The p-Frobenius number (or the generalized Frobenius number) for a numerical semigroup generated by $a_1, a_2, \ldots, a_k$ with $\gcd(a_1, a_2, \ldots, a_k) = 1$ is the largest integer that the number of solutions of the nonnegative integer combination of $a_1, a_2, \ldots, a_k$ is at most p. Note that the 0 The Frobenius number is the classical Frobenius number. Recent research results exist on these p-Frobenius numbers, and studies have mainly been conducted to define p-Apery sets and p-Normal semigroups to obtain p-Frobenius numbers as related characteristics or to find p-Frobenius of Affine semigroups.

Many research results have been conducted on the numerical semigroup itself, including theoretical or algorithmic techniques to save the numerical semiconductor itself, research on numerical semigroup with certain types of constraints, research on complexity for the numerical semiconductor, and research on various indicators given constraints on the concentration of the numerical semiconductor [17]-[26].

In recent years, there have been many attempts to find the algorithmic complexity of the problem of finding the Frobenius number. For example, in [27], for the equation

$$a_0x_0 + a_1x_1 + \ldots + a_nx_n = t \tag{1}$$

where $x_i \geq 0$ for all i, and
$a_0 \leq a_1 \leq \ldots \leq a_n$, $\gcd(a_0, a_1, \ldots, a_n) = 1$, the author studies pseudo-polynomial time algorithms when the running time is a function of a_0 or a_n.

The study on the Frobenius problem for various numerical semigroups is still ongoing. For example, for the sequence

$$a_1 = \sum_{j=0}^{n-1} b_j(a_i - a_{i-1} = ab^{i-2}, for\ every\ i \geq 2) \tag{2}$$

where $b \geq 2$ is a positive integer, we call $\{a_1, a_2, \ldots\}$ the generalized Repunit Numerical Semigroup [28] and the Frobenius number for

$$S_a(b,n) = \{c\_1a\_1 + c\_2a\_2 + \cdots \mid c_i \geq 0\ for\ all\ i\} \tag{3}$$

$$\mathrm{F}(S_a(b,n)) = \begin{cases} (n-1)(b^n - 1 - a) + a(\sum_{j=0}^{n-1} b_j)\ if\ a < b^n - 1 \\ b^n - 1 - a + a(\sum_{j=0}^{n-1} b_j)\ if\ a > b^n - 1 \end{cases} \tag{4}$$

So, the Frobenius number can be characterized to only two cases. There are some recent results about Frobenius problems that have been unsolved for a long time, which seem to be just simple numerical semigroups. For example, in [29], the authors solve the Frobenius problem for

$$\mathrm{A} = \{\mathrm{a}, \mathrm{a}+1, \mathrm{a}+2^2, \ldots, \mathrm{a}+k^2\} \tag{5}$$

where a, k is a fixed natural number and a > 2. After that, the authors in [30] generalized this result to solve the Frobenius problem for

$$\mathrm{A(a)} = \{\mathrm{a}, \mathrm{ha}+\mathrm{d}, \mathrm{ha}+b_2\mathrm{d}, \ldots, \mathrm{ha}+b_k\mathrm{d}\} \tag{6}$$

with some partial cases for a, h, and b_i.

Also, there are results about the Frobenius number generated by some binomial coefficients, such as [3]. There are two results: the Frobenius number for the numerical semigroups generated by triangular and tetrahedral numbers. In the Result and Discussion Section, we focus on the Frobenius number for the numerical semigroups generated by triangular numbers.

While researching estimating the Frobenius number for the above numerical semigroups, we are simultaneously working on computing the Frobenius number for numerical semigroups that have not yet been studied. In this sequel, we are proving the Frobenius numbers for various cases of numerical semigroups. If a predictive model can estimate the Frobenius number even without proofs, it might be more helpful for predictions and provide additional support for our proofs. Therefore, we decided to use a similar approach to predict the Frobenius number for the numerical semigroups we are researching.

In general, finding a Frobenius number involves some steps. First, we search for the smallest integer of sets that can equally represent the set formed by nonnegative integer combinations of elements of a given numerical semigroup. For a given numerical semigroup, we call it the minimal system of generators. Finding it has many advantages not only it is essentially included in the steps to find the Frobenius number, but when we count the elements, we can find that it is infinite despite the minimal system of generators can be finite. Once we define a reasonable minimal system of generators, we define what is called an Apery set. The Apery set of n in S, or Ap(S, n), is the set of all the smallest elements of a given numerical semigroup that are 0, 1,...,n-1 modulo n, respectively. Of course, the number of elements of Ap(S, n) is n, and it is known that the Frobenius number is the largest integer of Ap(S, n) minus n.

The process of finding the Frobenius number and other parameters such as genus, Pseudo-Frobenius number, type, etc. are also important, but we will not discuss it in this paper. In the above, in the part of finding the Frobenius number, it is especially important to find the Apery set, and the main difficulty is to find the shape of this set. Therefore, this paper describes how to estimate the Frobenius number without finding the Apery set and finding only the minimal system of generators. The advantage of this method is that it does not need to find the Apery set, and the limitation is that it may be difficult to guess the Frobenius number depending on the shape of the minimal system of generators.

Below, we describe the process of finding the Frobenius number for a set of two mutually prime natural numbers to help readers understand.

- Let S = {5,7,14}. Then S is a numerical semigroup because gcd(5,7,14) = 1.
- The set generated by no negative integer combination of the elements of S is <S> = {5a + 7b + 14c | a, b, c ≥ 0, and a, b, c are integers}.
- Since 14c can be replaced by 7(2c), we can rewrite <S> = {5a + 7b | a, b ≥ 0, and a, b are integers} and {5, 7} is a minimal system of generators of S.
- To obtain Ap(S, 5), we classify the nonnegative integers in <S>.
- All nonnegative integers congruent to 0 modulo 5 are 0, 5, 10, ... and all numbers are also the element of <S>.
- All nonnegative integers congruent to 1 modulo 5 are 1, 6, 11, ... and the integers greater than or equal to 21 are

elements of <S> because 21 = 3•7 and 21 + 5n = 5n + 3•7 for any nonnegative integer n.

- All nonnegative integers congruent to 2 modulo 5 are 2, 7, 12, ... and the integers greater than or equal to 12 are the elements of <S> because 12 = 5 + 7 and 12 + 5n = 5(n+1) + 7 for any nonnegative integer n.
- All nonnegative integers congruent to 3 modulo 5 are 3, 8, 13, ... and the integers greater than or equal to 28 are the elements of <S> because 28 = 4•7 and 28 + 5n = 5n + 4•7 for any nonnegative integer n.
- All nonnegative integers congruent to 4 modulo 5 are 4, 9, 14, ... and the integers greater than or equal to 14 are the elements of <S> because 14 = 2•7 and 14 + 5n = 5n + 2•7 for any nonnegative integer n.
- Therefore we can conclude that the Apery set of 5 in S is Ap(S, 5) = {0, 21, 12, 28, 14} and the Frobenius number for S is F(S) = max(Ap(S, 5)) − 5 = 28 − 5 = 23.

In this paper, we describe the method to get a formula for Frobenius number by estimating the asymptotic property of the Frobenius number using Linear regression. We introduce two numerical semigroups, one where the method of this paper applies well, and the other where the method of this paper does not apply well. The first one is

$$S_1 = \{ \binom{2p}{1}, \binom{2p}{2}, \dots, \binom{2p}{2p-1} \} \tag{7}$$

where p is an odd prime number. And the second one is

$$S_2 = \{ \binom{n+1}{2}, \binom{n+2}{2}, \dots \} \tag{8}$$

where n is any integer greater than 1.

The first case, S_1, by the well-known formula $\binom{n}{k} = \binom{n}{n-k}$, S_1 can be replaced by

$$S_1 = \{ \binom{2p}{1}, \binom{2p}{2}, \dots, \binom{2p}{p} \} \tag{9}$$

and we can observe that $\binom{2p}{1} = 2p, \binom{2p}{2} = p(2p-1)$ and for any positive integer k less than p, $\binom{2p}{k}$ is a multiple of p because k! does not divided by p. Therefore $\binom{2p}{k}$ is a nonnegative integer combination of $\binom{2p}{1}, and \binom{2p}{2}$ and we can conclude that a minimal system of generators is $\{\mathrm{p}\binom{2p}{1} + q\binom{2p}{2} + r\binom{2p}{p} \mid \mathrm{p.q.r\ are\ nonnegative\ integers}\}$ and we denote the set <S_1>.

The second case, S_2, we can rewrite:

$$< S_2 > = \{p\binom{n+1}{2} + q\binom{n+2}{2} + r\binom{n+3}{2} \mid p.q.r\ \ are\ nonnegative\ integers\} \tag{10}$$

because of the minimal system of generators of S_2. This paper [3] already provides the formula for the Frobenius number for S_2, but we assume that we do not know this formula. Instead, we only know the Frobenius numbers for several individual natural numbers. We will use linear regression to estimate the Frobenius number in general cases.

II. Materials and Method

The first case, $S_1 = \{ \binom{2p}{1}, \binom{2p}{2}, \dots, \binom{2p}{2p-1} \}$, we compute the Frobenius number for all the prime numbers less than 10,000. To obtain the general formula, we use the Linear Regression. In this study, we use the "pandas" library of the Python program, which can compute the constant and the linear coefficient. Because the exact Frobenius number is so large to compute, we pre-process the data to log-scale and we analyze them.

After completing the log-scale approximation, we exponentiated the generated function using linear regression to estimate the Frobenius number. From this, we made an estimation about the coefficient of the largest element in the nonnegative integer combination of the minimal system of generators, which represents the Frobenius number.

Fortunately, computing the coefficient of the second element of the minimal system of generators is so easy because this value is completely dependent to the Frobenius number and the coefficient of the largest element of the minimal system of generators.

For the second case $S_2 = \{\binom{n+1}{2}, \binom{n+2}{2}, \dots\}$, we compute the Frobenius number for all natural numbers $3 \leq \mathrm{n} \leq 1002$. After that, from the computed data, we estimate the Frobenius number for general n using the Linear Regression. In the process, we found that the data was split into two distinct functions, which were not predicted by a simple linear regression, but fortunately, both were defined as polynomial functions that depended on the remainder of n divided by 2, so it wasn't too difficult to solve the problem as long as we classified them correctly.

III. Results and Discussion

The following Fig. 1 shows the estimation of the log of the Frobenius number for

$$S_1 = \{ \binom{2p}{1}, \binom{2p}{2}, \dots, \binom{2p}{p} \} \tag{11}$$

using the Linear Regression for each odd prime p < 10000.

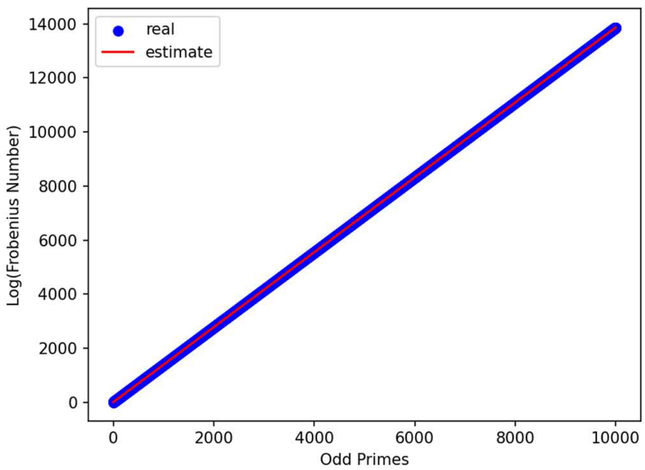

Fig. 1 Estimation of the log of Frobenius number for $\boldsymbol{S_1}$ for odd primes.

The estimated linear function is

$$\mathrm{f(x)} = 2.783 + 1.386\mathrm{x} \tag{12}$$

with the Regression Score 0.9999999961656982. Therefore, we can estimate that the Frobenius number is $e^{2.783+1.386p} \approx 16.17 \bullet e^{1.386p}$ for any sufficiently large prime number p.

By the following Asymptotic behavior of ln $\binom{2p}{p}$

$$\lim_{p\to\infty} \frac{\ln\binom{2p}{p}}{p} = 2\ln 2 \quad (13)$$

and $2\ln 2 \approx 1.386$, we can estimate that

$$F(S_1) = \Omega(\binom{2p}{p}). \quad (14)$$

To obtain the exact coefficient for $\binom{2p}{p}$ of $F(S_1)$, we compute the integer.

$$c_2 = max(t : F(S_1) - t\binom{2p}{p} > 0) \quad (15)$$

for each odd prime p < 1000. After that, we estimate c_2 using the Linear Regression.

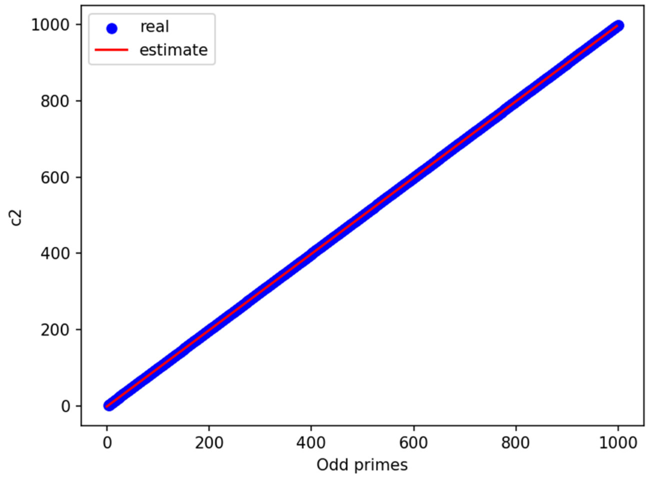

Fig. 2 Estimation of c_2 for odd primes

The estimated linear function is

$$f(x) = -0.999 + x \quad (16)$$

with the Regression score 1.0. Therefore, we can conclude that $t = p - 1$ for any sufficiently large prime p and we have:

$$F(S_1) = max(Ap(S, \binom{2p}{1})) - \binom{2p}{1} = b\binom{2p}{2} + (p - 1)\binom{2p}{p} - \binom{2p}{1} \quad (17)$$

and the last part is obtaining b, because:

$$b = \frac{F(S_1) - (p-1)\binom{2p}{p} + \binom{2p}{1}}{\binom{2p}{2}} \quad (18)$$

we compute the value b for odd primes p < 1000 and all values of b is 1. Therefore, we conclude that:

$$F(S_1) = \binom{2p}{2} + (p-1)\binom{2p}{p} - \binom{2p}{1} \quad (19)$$

and it is equal to

$$p(2p-3) + (p-1)\binom{2p}{p} \quad (20)$$

for any sufficiently large p.

The following Fig. 3 shows the estimation of the Frobenius number for

$$S_2 = \{\binom{n+1}{2}, \binom{n+2}{2}, \dots\} \quad (21)$$

using the Linear Regression.

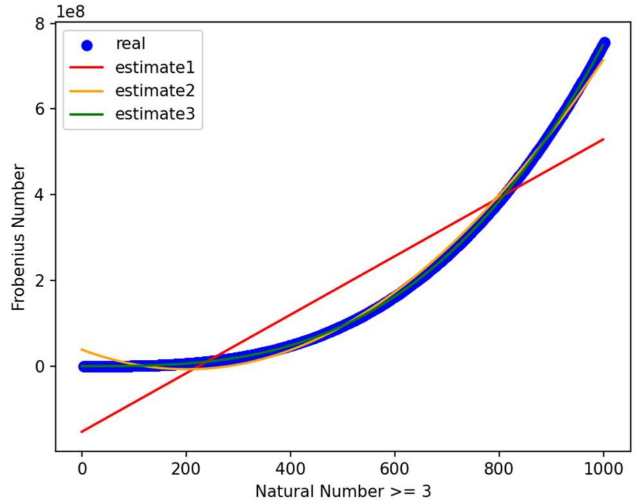

Fig. 3 Estimation of the Frobenius number for S_2 for natural numbers greater than or equal to 3.

The estimated function is

$$f(x) = -786.381 + 11.886x + 1.841\boldsymbol{x^2} + \mathbf{0.750}\boldsymbol{x^3} \quad (22)$$

with the Regression score 0.9999993782505251. This prediction is better than the prediction made using linear regression with a linear or quadratic function. Because the 1st and 2nd degree of f(x) do not seem to be clear to estimate the exact formula, we set.

$$g(n) = F(S_2) - \frac{3}{4}n^3 \quad (23)$$

and use the Linear Regression again.

Fig. 4 shows the g(x) estimation using the Linear Regression.

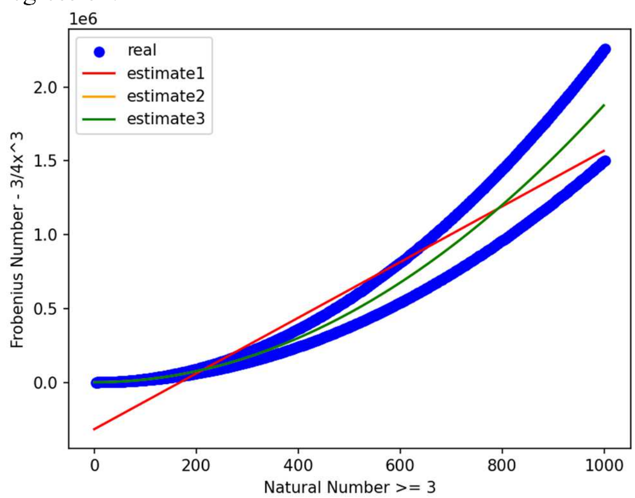

Fig. 4 Estimation of the Frobenius number for S_2 - $\frac{3}{4}x^3$ for natural numbers greater than or equal to 3.

Fig. 4 shows that g(x) cannot be estimated exactly because g(x) splits into two distinct functions. Fortunately, we can observe that the real values of g(x) have a pattern in which these values depend on the value of x modulo 2, as shown in Figs. 5 and 6.

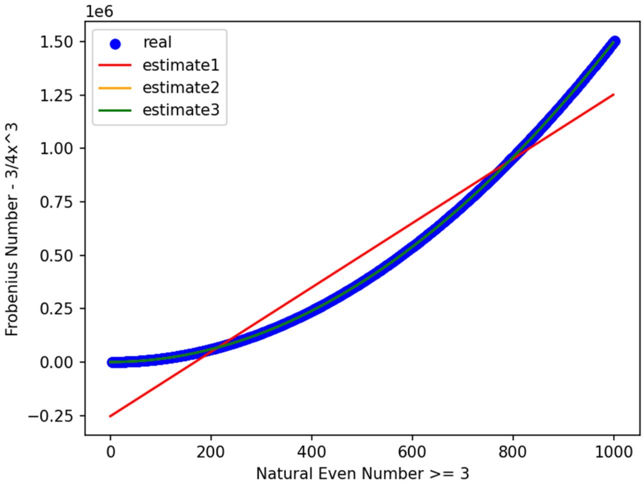

Fig. 5 Estimation of the Frobenius number for S_2 - $\frac{3}{4}x^3$ for natural even numbers greater than or equal to 3.

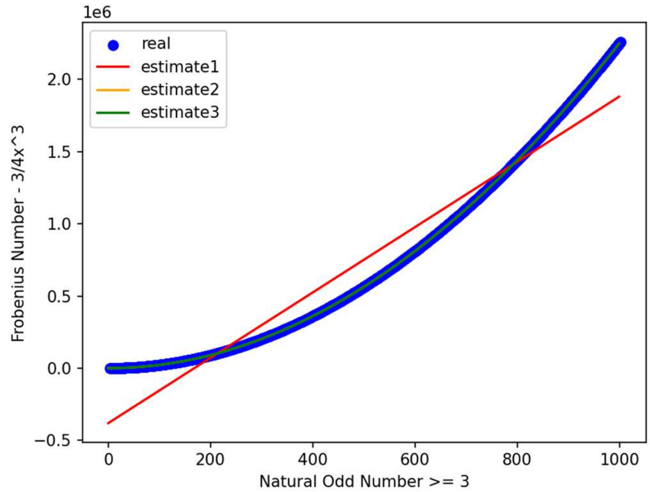

Fig. 6 Estimation of the Frobenius number for S_2 - $\frac{3}{4}x^3$ for natural odd numbers greater than or equal to 3.

As in Fig. 5 and Fig. 6, we can estimate g(n) as

$$g(n) = \begin{cases} -2.5 - 0.75n + 1.5n^2 \text{ if } n \text{ is odd} \\ -1 + 1.5n + 2.25n^2 \text{ if } n \text{ is even} \end{cases} \quad (24)$$

with both of the regression scores are exactly 1.0. Therefore, we can conclude that:

$$F\left(\binom{n+1}{2}, \binom{n+2}{2}, \dots\right) = \begin{cases} \frac{3n^3 + 6n^2 - 3n - 10}{4} \text{ if } n \text{ is odd} \\ \frac{3n^3 + 9n^2 + 6n - 4}{4} \text{ if } n \text{ is even} \end{cases} \quad (25)$$

using $F\left(\binom{n+1}{2}, \binom{n+2}{2}, \dots\right) = g(n) + \frac{3}{4}n^3$. This is exactly the same as the result of [3].

IV. Conclusion

As mentioned above, the Apery set provides much information about the numerical semigroup, including the Frobenius number. Still, the process of finding it requires a large amount of computation to account for all the incongruent elements for modulo n. If the shape of the numerical semigroup is complex, it becomes intractable even for computers. On the other hand, since the Frobenius number is in most cases determined by some function of n independent of modulo n, it was possible to approach the problem by making a function prediction, given enough data on individual Frobenius numbers. While these Frobenius numbers are estimates only, and need to be proved for clarity, it is safe to say that given the fact that the Frobenius number is a nonnegative integer combination of each element of a minimal system of generators, i.e., the sum over a discrete number of products, predicting the Frobenius number via linear regression will in effect give the correct answer to the Frobenius number.

A good topic for future research would be to create models that use data science techniques to predict computationally complex sets, such as the Apery set, which is at the core of the Frobenius problem. As mentioned above, this requires a lot more thought, as it requires ensuring that the numbers obtained are the smallest number represented by a combination of nonnegative integers, each congruent to a specific value for modulo n, rather than simply predicting a function. Still, it is expected that an accurate prediction model for the Apery set will be able to provide a solution to the Frobenius problem in most cases. Proof would still be required, but knowing and proving an almost exact answer would make the research much easier for the mathematician and would be a very nice tool to have.

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