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Estimating the Occurrence Rate for Alpha-Series Process in Rayleigh Distribution

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***Abstract*—The geometric process is sometimes appropriate for reliability and scheduling problems. Some previous studies suggested a possible alternative process that is alpha-series as to the geometric process when it decreases with time, as the decreasing geometric process shows that the expected number of events at an arbitrary time does not exist. In contrast, the expected number of events of the alpha-series process (ASP) exists at an arbitrary time under some conditions. In this paper, we assumed that the first arrival followed the Rayleigh distribution (RD). The modified moment estimator was proposed to estimate the alpha-series process parameters in the Rayleigh distribution and compare it with the maximum likelihood estimators. A simulation was conducted to compare the two estimators. The real-data application of intervals between successive failures of the Mosul Dam power station in Nineveh governorate in Iraq is provided to illustrate the results. When the initial occurrence time distribution is indicated to be RD, an estimate of the occurrence rate of an ASP is investigated in this study. Estimators are generated using modified moment (MM), and maximum likelihood (ML) approaches. According to the simulation study's findings, the MM estimator outperforms the ML estimator. In all cases, ASP with RD provides better data than the renewal process (RP) in real data sets. A test statistic has been devised to determine if the data conforms to an ASP.**

***Keywords*—Alpha-series process; Rayleigh distribution; maximum likelihood estimator; modified moment estimator; Monte Carlo simulation.**

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I. INTRODUCTION

Derive the statistical inference of the geometric process (GP) when the first arrival time is assumed to be Rayleigh distribution (RD). Applying the counting process from a statistical point of view is a common method of analyzing a data set with successive events' time. If successive times have a monotonous trend, the monotonous intensity function for the non-homogeneous Poisson process can be considered a possible method [1]. A more direct approach to modeling this data is to use a monotonous counting process in the alpha-series process [2].

Definition 1. The simple definition of the Renewal Process (RP) indicates that many stochastic processes can be described as RP. If $\{X(t)\}$ defined to be a sequence of non-negative i.i.d random variables representing the intervals between the occurrences of events observed from a counting process, RP is a possible approach for modeling this process. RP has two main parameters μ and σ^2 [3]:

$$\hat{\mu} = \bar{X} \tag{1}$$

$$\hat{\sigma}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2 \tag{2}$$

Definition 2. Alpha-Series Process (ASP) is one of the stochastic processes associated with the GP. It is one of the monotonous processes that its idea goes back to the researcher and its properties. Nevertheless, a regular linear connection may be discovered. Variables that lead to multicollinearity are the ones that must be explained and are tricky [4].

Let the stochastic process $\{N(t), t \geq 0\}$ represent the counting process and that $\{X_i\}$ represents times between event $(i-1)$ to event (i), so the counting process $\{N(t)\}$ is called ASP when there is αas a real number such as follows [5].

$$Y_i = i^{\alpha} X_i, i = 1,2, \ldots \tag{3}$$

are i.i.d. random variables with the distribution function F. The cumulative function of ASP is [6]:

$$F_i(x) = F_i(i^{\alpha} x) \forall i = 1,2, \ldots \tag{4}$$

By deriving the equation (4) with respect to$x$the probability density function of the ASP is obtained:

$$\frac{\partial F_i(x)}{\partial x} = f_i(x) = i^\alpha f_i(i^\alpha x_i) \quad (5)$$

Where $(i^\alpha x_i)$ represents an RP, a sequence of non-negative random variables i.i.d [7].

Some theoretical characteristics of the ASP [8]:

If $(\alpha > 0)$, then ASP is randomly decreasing that is,

$$X_i >_{st} X_{i+1} \forall i = 1,2,\dots$$

If $(\alpha < 0)$, then ASP is randomly increasing that is,

$$X_i <_{st} X_{i+1} \forall i = 1,2,\dots$$

If $(\alpha = 0)$, then ASP is an RP [9].

The expectation and variance $X_i$are:

$$E[X_i] = \frac{\mu}{i^\alpha}, i = 1,2,3,\dots \quad (6)$$

$$Var(X_i) = \frac{\sigma^2}{i^{2\alpha}}, i = 1,2,3,\dots \quad (7)$$

The parameters α, μ and σ^2 are important parameters for the ASP, given that knowing these parameters leads to knowledge of both the process's mean and variance and the process's general trend and power [10]. The statistical inference results of ASP were recently presented with the assumption that the random variable X_1 follows specific distributions [11], [12].

In this paper, estimating the occurrence rate of ASP, the first occurrence time to be Rayleigh is proposed. It can be as a possible alternative distribution. The Monte Carlo simulation's experimental results show the proposed distribution's favorable performance. The estimation methods used to estimate the time rate of occurrence of the ASP are the maximum likelihood and modified moment estimation methods [13]. The Mosul Dam power station analyzed the data set as part of applied research.

II. Materials and Methods

A. Rayleigh Distribution (RD)

The RD is a continuous probability distribution for nonnegative-valued random variables [14] introduced this distribution and discussed its characteristics. RD is applied in many fields, such as health, agriculture, biology, and other science. RD is a special case distribution of a Weibull with two parameters where the shape parameter equals two yields the following probability density function [15]:

$$f(x,y) = \begin{cases} \frac{x}{\lambda^2} exp\left(\frac{-x^2}{2\lambda^2}\right), x > 0 \\ 0, otherwies \end{cases} \quad (8)$$

with mean given the following equations:

$$E(X) = \lambda\sqrt{\frac{\pi}{2}} \quad (9)$$

and variance:

$$Var(X) = \frac{4-\pi}{2}\lambda^2 \quad (10)$$

The cumulative distribution function is given by:

$$F(x) = 1 - exp\left(\frac{-x^2}{2\lambda^2}\right) \quad (11)$$

B. Maximum Likelihood Method

Suppose that the set of data of ASP is $\{X_1, X_2, \dots, X_n\}$ with parameter (α) . X_1 is supposed to distribute as RD with parameter (λ). The likelihood function of ASP is as follows [16]:

$$L(\alpha,\lambda) = \frac{\prod_{i=1}^{n}(i^{2\alpha}x_i)}{(\lambda^2)^n} e^{-\sum_{i=1}^{n}(i^\alpha x_i)^2/2\lambda^2} \quad (12)$$

The log-likelihood function is expressed as follows:

$$ln\, L\,(\alpha,\lambda) = 2\alpha\sum_{i=1}^{n} ln\, i + \sum_{i=1}^{n} ln\, x_i - 2n\, ln\, \lambda - \frac{\sum_{i=1}^{n}(i^\alpha x_i)^2}{2\lambda^2} \quad (13)$$

Hence, deriving equation (13) for the parameters α and λ. Then likelihood function will be as follows:

$$\frac{\partial\, ln\, L(\alpha,\lambda)}{\partial\lambda} = \frac{-2n}{\lambda} + \frac{2\sum_{i=1}^{n}(i^\alpha x_i)^2}{2\lambda^3} = 0 \quad (14)$$

$$\frac{\partial\, ln\, L(\alpha,\lambda)}{\partial\alpha} = 2\sum_{i=1}^{n} ln\, i - \frac{2\sum_{i=1}^{n}(i^\alpha x_i)^2\, ln\, i}{2\lambda^2} = 0 \quad (15)$$

By solving equations (14), the parameter $\lambda$is found as:

$$\lambda = \sqrt{\frac{\sum_{i=1}^{n}(i^\alpha x_i)^2}{2n}} \quad (16)$$

Substitution of λ into equation (15) gives:

$$2\sum_{i=1}^{n} ln\, i - [\sum_{i=1}^{n} 2n(i^\alpha x_i)^2\, ln\, i][\sum_{i=1}^{n}(i^\alpha x_i)^2]^{-1} = 0 \quad (17)$$

Let $\hat{\alpha}_{ML}$and $\hat{\lambda}_{ML}$ referred to ML estimators of α and λ respectively. It is clear that an explicit form of the solution of equation (17) does not exist. Hence the numerical methods must be used as a solution to equation (17). Newton Raphson formula is used to obtain $\hat{\alpha}_{ML}$as follows [17, 18]:

$$\alpha_{n+1} = \alpha_n - \frac{f\, l(\alpha_n)}{f'\, l(\alpha_n)} \quad (18)$$

Where f is the objective function which is the equation (17). By Substituting $\hat{\alpha}_{ML}$in equation (16), the ML estimator of λ will be resulted such as:

$$\hat{\lambda}_{ML} = \sqrt{\frac{\sum_{i=1}^{n}\left(i^{\hat{\alpha}_{ML}} x_i\right)^2}{2n}} \quad (19)$$

Theorem: The joint distribution of the ML estimator is an asymptotically normal distribution (AN) with mean vector $(\alpha, \lambda)$and var-covariance matrix FI^{-1}, that is [19-21]:

$$\begin{pmatrix}\hat{\alpha}_{ML}\\ \hat{\lambda}_{ML}\end{pmatrix} \sim AN\left(\begin{pmatrix}\alpha\\ \lambda\end{pmatrix}, FI^{-1}\right) \quad (20)$$

Where FI^{-1}represents the inverse of the Fisher information matrix given as:

$$FI^{-1} = \begin{bmatrix} \frac{n}{4[n\sum_{i=1}^{n}(ln\, i)^2 - (\sum_{i=1}^{n} ln\, i)^2]} & \frac{\lambda\sum_{i=1}^{n} ln\, i}{4[n\sum_{i=1}^{n}(ln\, i)^2 - (\sum_{i=1}^{n} ln\, i)^2]} \\ \frac{\lambda\sum_{i=1}^{n} ln\, i}{4[n\sum_{i=1}^{n}(ln\, i)^2 - (\sum_{i=1}^{n} ln\, i)^2]} & \frac{\lambda^2\sum_{i=1}^{n}(ln\, i)^2}{4[n\sum_{i=1}^{n}(ln\, i)^2 - (\sum_{i=1}^{n} ln\, i)^2]} \end{bmatrix}$$

To derive the inverse Fisher information matrix, the second derivatives are taken from the log-likelihood function, which is equation (13) for (α, λ), we get [22]:

$$\frac{\partial^2\, ln\, L(\alpha,\lambda)}{\partial\alpha^2} = \frac{-2\sum_{i=1}^{n}(i^\alpha x_i)^2(ln\, i)^2}{\lambda^2} \quad (21)$$

$$\frac{\partial^2\, ln\, L(\alpha,\lambda)}{\partial\lambda^2} = \frac{2n}{\lambda^2} - \frac{3\sum_{i=1}^{n}(i^\alpha x_i)^2}{\lambda^4} \quad (22)$$

$$\frac{\partial^2 \ln L(\alpha,\lambda)}{\partial\alpha\ \partial\lambda} = \frac{2\sum_{i=1}^{n}(i^\alpha x_i)^2 \ln i}{\lambda^3} \quad (23)$$

Since $E(i^\alpha X_i) = \lambda\sqrt{\frac{\pi}{2}}$ and $E(i^\alpha X_i)^2 = 2\lambda^2$ the expected values of the second derivatives are obtained as:

$$E\left(\frac{-\partial^2 \ln L(\alpha,\lambda)}{\partial\alpha^2}\right) = 4\sum_{i=1}^{n}(\ln i)^2 \quad (24)$$

$$E\left(\frac{-\partial^2 \ln L(\alpha,\lambda)}{\partial\lambda^2}\right) = \frac{4n}{\lambda^2} \quad (25)$$

$$E\left(\frac{-\partial^2 \ln L(\alpha,\lambda)}{\partial\alpha\,\partial\lambda}\right) = \frac{-4\sum_{i=1}^{n}\ln i}{\lambda} \quad (26)$$

These are the components of theFI matrix as follows:

$$FI = \begin{bmatrix} 4\sum_{i=1}^{n}(\ln i)^2 & \frac{-4\sum_{i=1}^{n}\ln i}{\lambda} \\ \frac{-4\sum_{i=1}^{n}\ln i}{\lambda} & \frac{4n}{\lambda^2} \end{bmatrix} \quad (27)$$

are its inverse are given by:

$$FI^{-1} = \begin{bmatrix} v_{11} & v_{12} \\ v_{12} & v_{22} \end{bmatrix} \quad (28)$$

$$FI^{-1} = \frac{1}{|FI|}adj(FI) \quad (29)$$

After making some simplifications, the matrix determined is as follows [23]:

$$|FI| = \frac{16}{\lambda^2}\left[n\sum_{i=1}^{n}(\ln i)^2 - \left(\sum_{i=1}^{n}\ln i\right)^2\right] \quad (30)$$

The components of the inverse of FImatrix as follows:

$$v_{11} = \frac{n}{4\left[n\sum_{i=1}^{n}(\ln i)^2 - \left(\sum_{i=1}^{n}\ln i\right)^2\right]} \quad (31)$$

$$v_{12} = v_{21} = \frac{\lambda\sum_{i=1}^{n}\ln i}{4\left[n\sum_{i=1}^{n}(\ln i)^2 - \left(\sum_{i=1}^{n}\ln i\right)^2\right]} \quad (32)$$

$$v_{22} = \frac{\lambda^2\sum_{i=1}^{n}(\ln i)^2}{4\left[n\sum_{i=1}^{n}(\ln i)^2 - \left(\sum_{i=1}^{n}\ln i\right)^2\right]} \quad (33)$$

Corollary: The marginal asymptotic distributions of the ML estimators of the parameters α and λ are:

$$\hat{\alpha}_{ML} \sim AN\left[\alpha, \frac{n}{4\left[n\sum_{i=1}^{n}(\ln i)^2 - \left(\sum_{i=1}^{n}\ln i\right)^2\right]}\right] \quad (34)$$

$$\hat{\lambda}_{ML} \sim AN\left[\lambda, \frac{\lambda^2\sum_{i=1}^{n}(\ln i)^2}{4\left[n\sum_{i=1}^{n}(\ln i)^2 - \left(\sum_{i=1}^{n}\ln i\right)^2\right]}\right] \quad (35)$$

Moreover, $H_0: \alpha = 0$vs $H_1: \alpha \neq 0$ can be tested by using:

$$R = \frac{\hat{\alpha}_{ML}}{\sqrt{\frac{n}{4\left[n\sum_{i=1}^{n}(\ln i)^2 - \left(\sum_{i=1}^{n}\ln i\right)^2\right]}}} \quad (36)$$

Where $\hat{\alpha}_{ML}$ can be evaluated from equation (18). Under the null hypothesisH_0, by Slutsky theorem from equation (34) i.e.,$R \sim AN(0,1)$. If $\alpha = 0$ then the dataset agrees with RP [24].

C. Modified Moment Method

In this section, the MM estimator of the ASP parameters is obtained by using the nonparametric estimate of the parameterα, which is a commonly used method for ASP. The researchers derived a nonparametric estimator for the parameter αas the follows [25]:

$$\hat{\alpha}_{NP} = \frac{\sum_{i=1}^{n}\ln X_i\sum_{i=1}^{n}\ln i - n\sum_{i=1}^{n}\ln i\ln X_i}{n\sum_{i=1}^{n}(\ln i)^2 - \left(\sum_{i=1}^{n}\ln i\right)^2} \quad (37)$$

By substituting equation (37) in equation (3) we obtain:

$$\hat{y}_i = i^{\hat{\alpha}_{NP}}X_i, i = 1,2,\dots,n \quad (38)$$

Let $\{X_1, X_2, \dots, X_n\}$ be a random sample from the ASP with parameter αand the first failureX_1follows the RD,$X_1 \sim R(\lambda)$. The parameter αis supposed to be the estimation of non-parametric by equation (37), the first sample moment m_1 is calculated by [26-28]:

$$m_1 = \frac{1}{n}\sum_{i=1}^{n}\hat{y}_i = \frac{1}{n}\sum_{i=1}^{n}i^{\hat{\alpha}_{NP}}X_i \quad (39)$$

to calculate the first population moment for RD, we need to find the E(X) which is denoted by the $\mu_1$can be easily written as [29, 30]:

$$\mu_1 = \lambda\sqrt{\frac{\pi}{2}} \quad (40)$$

by equating the first sample moment with the first population moment, we obtain the parameter of the RD [31, 32]:

$$\hat{\lambda}_{NP} = \sqrt{\frac{2}{\pi}\frac{1}{n}\sum_{i=1}^{n}i^{\hat{\alpha}_{NP}}X_i} \quad (41)$$

III. Results and Discussion

This section introduces some Monte Carlo simulation studies for comparing the efficiency of ML and MM estimators for the parameters α and λ. In the simulation study, different sample sizes were chosen $n = 30,50$. For each case, the RD parameter is taken $\lambda = 0.5,1$, and the exponent of the process is $\alpha = 0.5, 0.8, -0.5$. The experiment was repeated 1000 times for each case. The mean square error MSE is used to measure the performance of the ML and MM estimators. According to the results presented in Tables 1-2, when the sample sizes increase, the MSE values decrease for estimators of$\alpha$and λ. Also, the MM estimators of $\alpha$and λoutperform the corresponding ML estimator depending on the MSE value for all cases.

TABLE I
THE SIMULATED MSE OF THE ESTIMATOR α AND, WHEN$\lambda = 0.5$

MSE($\hat{\lambda}$)	MSE($\hat{\alpha}$)	Method	n	α
0.0770	0.0406	ML	30	0.5
0.0376	0.0359	MM		
0.0619	0.0117	ML	50	
0.0073	0.0033	MM		
0.1069	0.0302	ML	30	0.8
0.0277	0.0210	MM		
0.0320	0.0138	ML	50	
0.0081	0.0015	MM		
0.1364	0.0317	ML	30	-0.5
0.0233	0.0257	MM		
0.0646	0.0101	ML	50	
0.0128	0.0063	MM		

TABLE II

THE SIMULATED MSE OF THE ESTIMATOR α AND , WHEN $\lambda = 1$

MSE($\hat{\lambda}$)	MSE($\hat{\alpha}$)	Method	n	α
1.0157	0.0311	ML	30	0.5
0.0510	0.0188	MM		
0.4129	0.0190	ML	50	
0.0489	0.0169	MM		
1.9255	0.0675	ML	30	0.8
0.1694	0.0496	MM		
0.4446	0.0154	ML	50	
0.0219	0.0037	MM		
0.1448	0.0385	ML	30	-0.5
0.0589	0.0196	MM		
0.1343	0.0245	ML	50	
0.0168	0.0125	MM		

A. Application to a Real Data Set

In order to verify the rate of the occurrence of the ASP in an RD proposed in this paper in a real-life context, a real Mosul Dam dataset was used to demonstrate data analysis and estimation procedures. This data set consists of two units, the first unit has 58 observations, and the second unit has 62 observations. These two data sets show the intervals between successive failures of the Mosul Dam power station in Nineveh Governorate in Iraq.

For a dataset $\{X_1, X_2, \dots, X_n\}$, the fitted values of observations are:

$$\hat{X}_i = \begin{cases} \hat{\mu}_{ML} i^{-\hat{\alpha}_{ML}} & by an ASP with ML estimators \\ \hat{\mu}_{MM} i^{-\hat{\alpha}_{MM}} & by an ASP with MM estimators \\ \bar{X}_n & by a renwal process \end{cases} \tag{42}$$

Let $S_k = X_1, X_2, X_3, \dots X_n$, $k = 1,2,\dots,n$. Then a fitted value of S_k is $\hat{S}_k = \sum_{j=1}^{k} \hat{X}_j$. To evaluate the performance of ASP with the estimator's ML and MM, and RP for the dataset, the plot of S_k and $\hat{S}_k$ against k, $k = 1,2,\dots,n$, and MSE can be used.

$$MSE = \frac{1}{n} \sum_{k=1}^{n} \left(X_k - \hat{X}_k\right)^2 \tag{43}$$

To test whether or not the data conforms to an ASP, and to solve this problem we assume that we have a process $\{X_i, i = 1,2,\dots,n\}$, and we assume that:

$$y_i = i^{\alpha} x_i, i = 1,2,\dots,n \tag{44}$$

By taking the logarithm for equation (44), we have:

$$\ln y_i = \alpha \ln i + \ln x_i, i = 1,2,\dots,n \tag{45}$$

Since y_i is an independent random variable, the simple linear regression model can be used as:

$$\ln x_i = \gamma - \alpha \ln i + e_i, i = 1,2,\dots,n \tag{46}$$

If the exponential residuals are distributed as RD, the dataset $\{X_1, X_2, \dots, X_n\}$ can be modelled with RD. The residuals are obtained:

$$\hat{e}_i = \ln x_i - \hat{\gamma} + \hat{\alpha}_{MM} \ln i \tag{47}$$

Where: $\hat{\gamma} = \frac{\sum_{i=1}^{n} \ln i \sum_{i=1}^{n} \ln i \ln X_i - \sum_{i=1}^{n} (\ln i)^2 \sum_{i=1}^{n} \ln X_i}{\left(\sum_{i=1}^{n} \ln i\right)^2 - n \sum_{i=1}^{n} (\ln i)^2}$

The ordered exponential residuals $\exp(\hat{e}_i)$ against the quantiles of the RD are plotted in the Q-Q plot. Fig. 1 and 2 show that the data points are not very different from the straight line. Thus, we can conclude that the underlying distribution of the first-unit and second-unit data of Mosul Dam corresponds to the Rayleigh distribution.

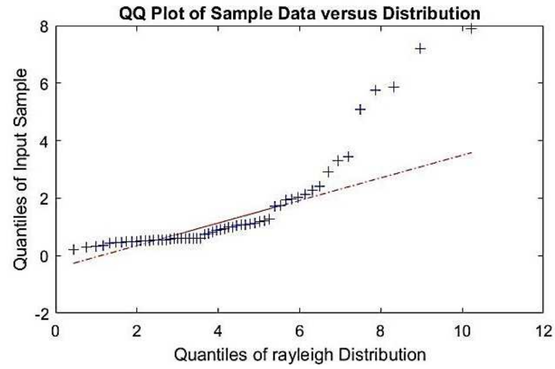

Fig. 1 Q-Q plot for the first unit's data of Mosul Dam.

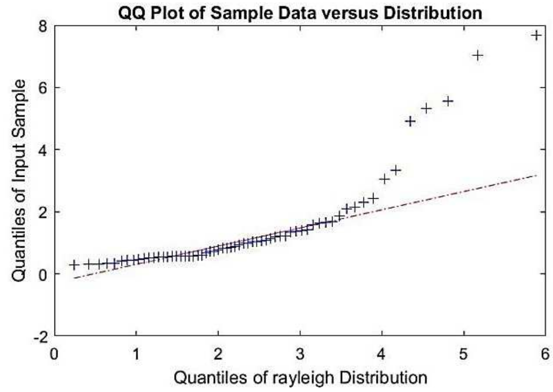

Fig. 2 Q-Q plot for the second unit's data of Mosul Dam.

From Fig.3, the times of failure for the first unit of Mosul Dam data and its fitted times against the number of failures are plotted by ASP with the estimator's ML, MM, and RP for the dataset, respectively. Note that ASP is a better fit for data than RP. This also supports $R - test$ in equation (36) where $R = 6.9626$ and the relevant p-value $= 1.6700 \times 10^{-12}$. This R-testdetects that the data follows the ASP. Table 3 shows an estimation of the α and λ parameters using MM and ML estimators, as well as includes the MSE values for the Mosul Dam. Moreover, it can be observed from table 3 that ASP with MM estimators is more reasonable than ASP with ML estimators.

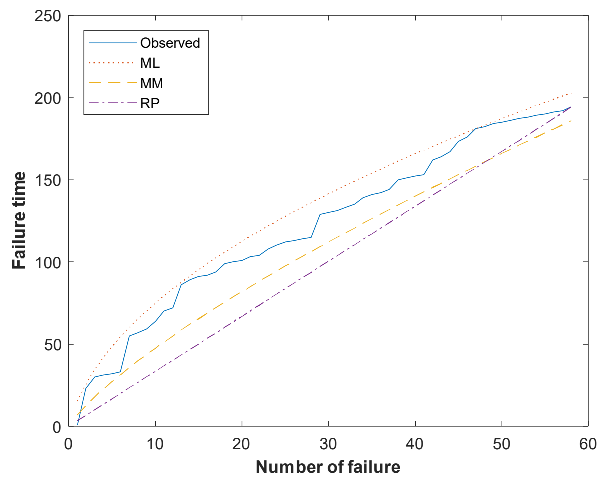

Fig. 3 Plotting the number of failures against S_i and $\hat{S}_i$ for the first unit of Mosul Dam.

TABLE III
ESTIMATION OF PARAMETERS AND MSE FOR THE FIRST UNIT'S DATA OF MOSUL DAM.

Method	$\hat{\alpha}$	$\hat{\lambda}$	MSE
MM	0.2539	5.4714	18.3122
ML	0.5131	81.5604	19.0716
RP	0	3.3448	20.1570

From Fig.4, the times of failure for the second unit of Mosul Dam data and its fitted times against the number of failures are plotted by ASP with the estimator's ML, MM, and RP for the dataset, respectively. Note that ASP is a better fit for data than RP. This also supports $R - test$in equation (36) where R $= 6.2570$ and the relevant $p - value = 1.9627 \times 10^{-10}$. This R-test detects that the data follows the ASP. Table 4 shows an estimation of the $\alpha$and λ parameters using MM and ML estimators, as well as includes the MSE values for the Mosul Dam. Moreover, it can be observed from table 4 that ASP with MM estimators is more reasonable than ASP with ML estimators.

TABLE IV
ESTIMATION OF PARAMETERS AND MSE FOR THE SECOND UNIT'S DATA ON THE MOSUL DAM

Method	$\hat{\alpha}$	$\hat{\lambda}$	MSE
MM	0.3564	8.4123	17.1684
ML	0.4437	77.4313	17.2579
RP	0	3.6613	19.9982

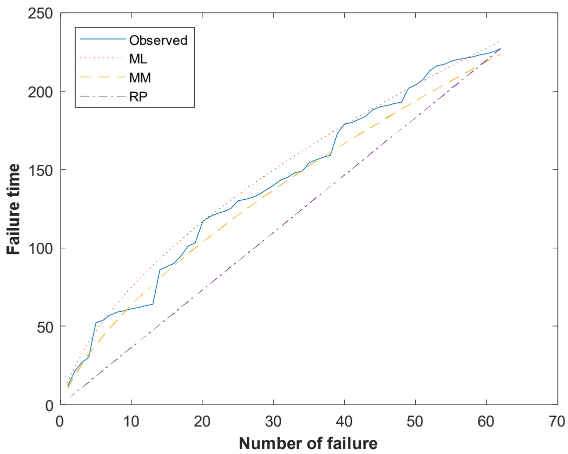

Fig. 4 Plotting the number of failures against $S_i$and $\hat{S}_i$ for the second unit of Mosul Dam.

IV. CONCLUSION

In this paper, an estimate of the occurrence rate of an ASP is studied when the first occurrence time distribution is suggested to be RD. Estimators $\alpha$and $\lambda$are obtained using MM and ML methods. The simulation study results showed that the MM estimator outperformed the ML estimator. In real data set applications, ASP with RD offers better data than RP in both applications. A test statistic has been developed to test whether the data conforms to an ASP.

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