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The Kumaraswamy Distribution: Statistical Properties and Application

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***Abstract*—Modeling and analyzing lifetime data is an important aspect of statistical work in various scientific and technological fields such as medicine, engineering, insurance, and finance. The modeling and analysis of lifetimes is an important aspect of statistical work in various scientific and technological fields. In recent years, inverted Kumaraswamy distribution has been used quite effectively to model many lifetime data. The most broadly applied statistical distribution is Kumaraswamy distribution in hydrological problems and many natural phenomena. The Kumaraswamy distribution (KD) is widely applied for modeling data in practical domains, such as medicine, engineering, economics, and physics. The present work proposes the Bayesian estimators of KD parameters through the use of type-II censoring data in this research the problem to estimate the unknown parameters of Kumaraswamy distribution with two parameters θ and λ, these estimates are a maximum likelihood of ordered observation and the Bayesian for the parameter of the Kumaraswamy distribution (KUD) depended on ranked set sampling (RSS) techniques. Both the simulated are inserted into real-life data sets and are considered to make a comparison between the estimation based on Maximum Likelihood estimators and Bayesian Estimation methods based on (RSS) techniques. For comparison purposes, we employed (100) mean square error and the criteria like AICC (Akaike information corrected criterion). Finally, the importance and flexibility of the new model of real data set are proved empirically.**

***Keywords*— Kumaraswamy distribution; Bayes estimation; reliability analysis; failure function; quantile function; order statistics.**

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I. INTRODUCTION

Originally the Kumaraswamy probability distribution was proposed by Poondi Kumaraswamy in 1980. The Kumaraswamy double bounded distribution is denoted by KUD (θ,λ) on the interval (0, 1), The Kumaraswamy is similar to the beta distribution but has the key advantage of closed from cumulative distribution function (CDF), has its probability density function (pdf) for Kumaraswamy with two parameters θ> 0 and λ> 0 is [1], [2].

$$f(x) = \theta^{\theta-1}(1-x^{\theta})^{-1} \qquad I(0\leq x\leq 1) \qquad (1)$$

and cumulative distribution function (cdf) for Kumaraswamy with two parameters θ> 0 and λ> 0 is as follows:

$$F(x) = 1-(1-x^{\theta})^{\lambda} \qquad (2)$$

Special cases of three parameter distribution with a density of Kumaraswamy distributions are as follows:

$$f(x) = \theta/B(\gamma,\lambda)\ \ x\hat{}(\gamma\hat{}(\theta-1)\ )\ (1-x\hat{}\theta\ )\hat{}(\lambda-1),(0\leq x\leq 1),\theta\ and\ \lambda>0 \qquad (3)$$

$$f(x) = \frac{\theta}{B(\gamma,\lambda)}\ x^{\gamma\theta-1}\,(1-x^{\theta})^{\ \lambda-1},(0\leq x\leq 1),\theta\ and\ \lambda\ >0$$

The present study will provide a mathematical formulation of the Kumaraswamy distribution and some of its properties. The research is organized as follows: In section 2, relationships with other distributions

Sections 3 and 4 are devoted to discussing the reliability analysis and various statistical properties of the (KUD). Moreover, the method of random number generation of the (KUD) and quintile's function, median are described in section 5. Further, in section 6, estimation using ranked set sampling (RSS) techniques by applying the method of maximum likelihood of ordered observation estimate and the Bayesian estimate are provided, respectively. Finally, in section 7, Monte Carlo simulation is used to construct the comparisons between estimates. The results are applied to real data. Finally, the research finishes with the conclusions.

Figure 1 shows some of the shapes in the pdf of a Kumaraswamy distribution for selected values of the parameters ($\theta = a$) and (λ=b)[3].

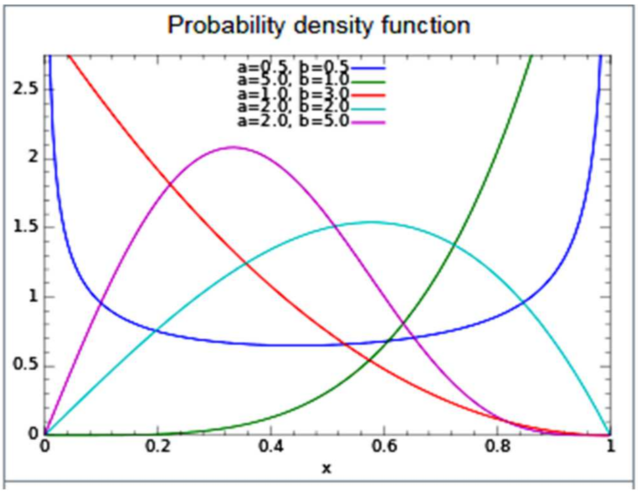

Fig. 1 The pdf's different Kumaraswamy distributions

Figure 1 gives us a detailed description of the different values parameters of the density function.

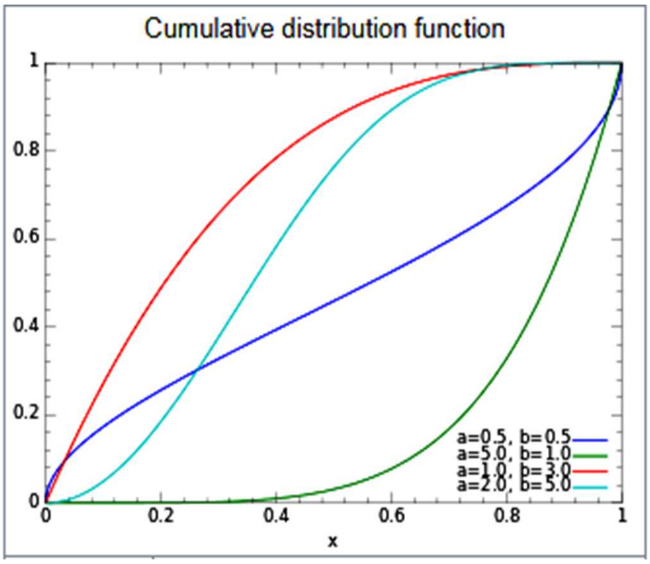

Fig 2. cdf's of different Kumaraswamy distributions

Figure 2 shows that the cumulative density function is an increasing function with different values of the parameters[3]

II. MATERIAL AND METHOD

This reliability discusses the reliability function, failure rate, and reverse failure rate of the (KUD) [4.]

A. Reliability Function

It may be defined as a probability that the item does not fail before sometimes t. It is denoted R(x). The reliability function can be mathematically obtained [5], [6].
R(x) =1-F(x)

$$R(t,\theta,\lambda)=(1-t^{\theta})^{\lambda},\ (0\le t\le 1),\theta \text{ and } \lambda>0 \quad (4)$$

B. Failure Function

It can be derived as the relation between the probability density function and the reliability function. It is denoted $h(t)=\frac{f(t)}{1-F(t)}$ and is given as [7].

$$h(t,\theta,\lambda)=\frac{\theta\ \lambda t^{\theta-1}}{(1-t^{\theta})},(0\le t\le 1),\theta \text{ and } \lambda>0 \quad (5)$$

C. Reverse Failure Function

The function rate is also an important quantity that characterizes life phenomena. It is given as [5], [8], [9]:

$$\varphi(x)=\frac{f(x)}{R(x)}=\frac{\theta\ \lambda x^{\theta-1}}{(1-x^{\theta})},(0\le x\le 1),\theta \text{ and } \lambda>0 \quad (6)$$

D. Statistical Properties to KUD Distribution

Kumaraswamy's distribution discusses moments, expected, and variance [10].

$$E(X^r)=\theta\,\lambda\,\frac{\Gamma\left(1+\frac{r}{\theta}\right)\Gamma(\lambda)}{\Gamma(1+\frac{r}{\theta}+\lambda)} \quad (7)$$

Especially we have

$$E(X)=\frac{\lambda\Gamma\left(1+\frac{1}{\theta}\right)\Gamma(\lambda)}{\Gamma\left(1+\frac{1}{\theta}+\lambda\right)} \quad (8)$$

$$Var(X)=\frac{\lambda\Gamma\left(1+\frac{2}{\theta}\right)\Gamma(\lambda)}{\Gamma\left(1+\frac{2}{\theta}+\lambda\right)}-\left(\frac{\lambda\Gamma\left(1+\frac{1}{\theta}\right)\Gamma(\lambda)}{\Gamma\left(1+\frac{1}{\theta}+\lambda\right)}\right)^2 \quad (9)$$

E. The Quantile Function, Median and Generating a Random Number

Can obtain the quantile function, median and generating random number KUD distribution [7], [11].

1) The quantile function, median: The quantile x_q Kumaraswamy distribution is the real solution of the equation [12], [13], [14].

$$X_q=\left[1-\ (1-x^{\frac{1}{\lambda}})\right]^{\frac{1}{\theta}} \quad (10)$$

The median of the distribution is calculated as:

$$X_{0.5}=\left[1-0.5^{\frac{1}{\lambda}}\right]^{\frac{1}{\theta}} \quad (11)$$

2) Random Number Generation: The method of inversion for Kumaraswamy distribution is generated as [15]:

$$1-(1-x^{\theta})^{\lambda}=u$$

and (u ~ U (0, 1)). After simplification, this yields as follows:

$$X=\left[1-\ (1-u^{\frac{1}{\lambda}})\right]^{\frac{1}{\theta}} \quad (12)$$

Equation (12), it can generate random numbers when the parameters θ,λ are known.

F. Estimation Using Ranked Set Sampling (RSS) Techniques

1) Maximum Likelihood Method: To estimate the unknown parameters of (KUD) use the technique of maximum likelihood by[16], [17], [7]:

Let $X_{(i:r)}j$; $0<X_{(i:r)}j<1, i=1,\dots,r$ and $j=1,\dots,w$ is a ranked set sample with sample size n = rw, from the Kumaraswamy distribution, where r set size, w number of cycles. For simplification purposes X (i:r) j, will be denoted as X ij. The pdf of the random variables X ij is given by [18], [19], [20]:

$$g(X_{ij})=\frac{r!}{(i-1)!\,(r-i)!}\,f(X_{ij})\,[F(X_{ij})]^{i-1}[1-F(X_{ij})]^{r-1}$$

which is in the case of the (KUD) distribution, will be as follows:

$$g(X_{ij}) = \frac{r!}{(i-1)!\,(r-i)!}\,\theta\lambda\, X_{ij}{}^{\theta-1}\left(1-X_{ij}{}^{\theta}\right)^{\lambda(r-i+1)-1} \times \left[1-\left(1-X_{ij}{}^{\theta}\right)^{\lambda}\right]^{i-1} \quad (13)$$

Then the likelihood function of θ $and\, \lambda$ the observed sample is given

$$L(data,\theta,\lambda) = K \prod_{j=1}^{w} \prod_{i=1}^{r} \Big(\theta\lambda\, X_{ij}{}^{\theta-1}\big(1 - X_{ij}{}^{\theta}\big)^{\lambda(r-i+1)-1}\left[1-\left(1-X_{ij}{}^{\theta}\right)^{\lambda}\right]^{i-1} \times \left[1-\left(1-X_{ij}{}^{\theta}\right)^{\lambda}\right]^{i-1}\Big) \quad (14)$$

The log-likelihood function given by

$$L(data,\theta,\lambda) = log\, log\, K + r\, w\, log\, log\, \theta + r\, w\, log\, log\, \lambda + (\theta-1)\sum_{j=1}^{w}\sum_{i=1}^{r} log\, log\, X_{ij} + \big(\lambda(r-i+1)-1\big)\sum_{j=1}^{w}\sum_{i=1}^{r} log\left(1-X_{ij}{}^{\theta}\right) + (i-1)\sum_{j=1}^{w}\sum_{i=1}^{r}\left[1-\left(1-X_{ij}{}^{\theta}\right)^{\lambda}\right] \quad (15)$$

where K is constant and Differentiating the log-likelihood function in (14) concerning θ and λ one can obtain [21]

$$\frac{\partial L}{\partial \lambda} = \frac{rw}{\lambda} + (r-i+1)\sum_{j=1}^{w}\sum_{i=1}^{r} log\left(1-X_{ij}{}^{\theta}\right) + (i-1)\sum_{j=1}^{w}\sum_{i=1}^{r}\frac{\left(1-X_{ij}{}^{\theta}\right)^{\lambda} log\left(1-X_{ij}{}^{\theta}\right)}{1-\left(1-X_{ij}{}^{\theta}\right)^{\lambda}} \quad (16)$$

and

$$\frac{\partial L}{\partial \theta} = \frac{rw}{\theta} + \sum_{j=1}^{w}\sum_{i=1}^{r} log\, log\, X_{ij} + \big(\lambda(r-i+1)-1\big)\sum_{j=1}^{w}\sum_{i=1}^{r}\frac{X_{ij}{}^{\theta} log[X_{ij}]}{\left(1-X_{ij}{}^{\theta}\right)} + \big(\lambda(r-i+1)-1\big)\sum_{j=1}^{w}\sum_{i=1}^{r} log\left(1-X_{ij}{}^{\theta}\right) + (i-1)\sum_{j=1}^{w}\sum_{i=1}^{r}\left[1-\left(1-X_{ij}{}^{\theta}\right)^{\lambda}\right] (i-1)\sum_{j=1}^{w}\sum_{i=1}^{r}\frac{X_{ij}{}^{\theta}\left(1-X_{ij}{}^{\theta}\right)^{\lambda-1} log[X_{ij}]}{1-\left(1-X_{ij}{}^{\theta}\right)^{\lambda}} \quad (17)$$

Equating the derivatives (16) and (17) to zero, one can obtain the ML estimator of the parameter λ, which is given by

$$\hat{\lambda}_{ML} = \frac{-rw}{\left[\sum_{j=}^{w}\sum_{i=1}^{r}\log\left(1-X_{ij}{}^{\theta}\right)+(i-1)\right]} \quad (18)$$

and by substituting (18) in (17), the ML estimate of the parameter θ is obtained numerically by applying any iteration procedure. For example, the estimation and credible interval of R(x) and h(x) are given in Section 3 [22], [23].

$$\hat{R}_{ML}(x) = \left(1-x^{\hat{\theta}}\right)^{\hat{\lambda}}, (0 \le x \le 1), \hat{\theta}\ and\ \hat{\lambda} > 0 \quad (19)$$

$$\hat{h}_{ML}(x) = \frac{\hat{\theta}\,\hat{\lambda} x^{\hat{\theta}-1}}{\left(1-x^{\hat{\theta}}\right)}, (0 \le x \le 1), \hat{\theta}\ and\ \hat{\lambda} > 0 \quad (20)$$

The asymptotic variances, and covariance of the MLEs, $\hat{\theta}$ $and\, \hat{\lambda}$, are known by the entries of the inverse to the Fisher information matrix $\frac{\partial L}{\partial \lambda} = \frac{rw}{\lambda} + (r-i+1)\sum_{j=1}^{w}\sum_{i=1}^{r} log\left(1-X_{ij}{}^{\theta}\right) + (i-1)\sum_{j=1}^{w}\sum_{i=1}^{r}\frac{\left(1-X_{ij}{}^{\theta}\right)^{\lambda} log\left(1-X_{ij}{}^{\theta}\right)}{1-\left(1-X_{ij}{}^{\theta}\right)^{\lambda}}$ $I_{ij} = E\left[\frac{-\partial^2 l(\emptyset)}{\partial\emptyset_i\partial\emptyset_j}\right]$, i=1,2 and $\emptyset = (\emptyset_1\emptyset_2) = (\theta,\lambda)$, Therefore, The asymptotic variance-covariance matrix of the ML estimates for the parameters θ and λ [24], [15]:

$$\hat{I}^{-1} = \left[\widehat{var}(\hat{\theta}_{ML})\ \widehat{cov}(\hat{\theta}_{ML},\hat{\lambda}_{ML})\ \widehat{cov}(\hat{\theta}_{ML},\hat{\lambda}_{ML})\ \widehat{var}(\hat{\lambda}_{ML})\right] = \frac{1}{|I|}\left[-\frac{\partial^2 L}{\partial\lambda^2}\ \frac{\partial^2 L}{\partial\theta\partial\lambda}\ \frac{\partial^2 L}{\partial\lambda\partial\theta} - \frac{\partial^2 L}{\partial\theta^2}\right]_{\hat{\theta}_{ML},\hat{\lambda}_{ML}} \quad (21)$$

The asymptotic normality of the ML estimates can by using (21), θ , λ can be found in confidence intervals, respectively

$$\hat{\theta}_{ML} \mp Z_{\frac{(1-r)}{2}}\sqrt{\widehat{var}(\hat{a})} \quad , \hat{\lambda}_{ML} \mp Z_{\frac{(1-r)}{2}}\sqrt{\widehat{var}(\hat{\lambda})}$$

where $Z_{\frac{(1-r)}{2}}$ is the upper αth quantile of the standard normal distribution. Using the language R can easily compute the Hessian matrix, and its inverse and further the standard errors and asymptotic confidence intervals [25], [26].

2) Bayesian Estimation: The Bayes estimators of parameters θ and λ denoted by $\hat{\theta}_{Bayes}$ ' $\hat{\lambda}_{Bayes}$ respectively, are obtained under the assumption that θ and λ are independent random variables with prior distributions Gamma (θ_1,λ_1) and Gamma (θ_2,λ_2) respectively with pdf's [3]:

$$\pi_1(\theta) = \frac{(\lambda_1)^{\theta_1}}{\Gamma(\theta_1)}\,\theta^{\theta_1-1}\,e^{-\lambda_1\theta} \quad (22)$$

and

$$\pi_2(\lambda) = \frac{(\lambda_2)^{\theta_2}}{\Gamma(\theta_2)}\,\lambda^{\theta_2-1}\,e^{-\lambda_1\lambda} \quad (23)$$

Where θ and λ > 0, and $(\theta_1,\theta_2,\lambda_1,\lambda_1) > 0$. The Bayes estimators of the shape parameters θ and λ denoted by and respectively, Let θ and λ be independent random variables with prior distributions given in Equations (22) and (23). Based on these assumptions and the likelihood function presented from Equation (14), the joint density of the data, θ and λ can be as [20], [27], [28]:

$$L(data;\theta,\lambda)\,\pi(\theta)\,\pi(\lambda) \quad (24)$$

$$\therefore L(data,\theta,\lambda) = K_1\,\Psi \quad (25)$$

Therefore, a posterior joint density to data θ and λ given the data can be obtained by[29], [30].

$$\pi_{Bayes}\left(\frac{\theta,\lambda}{data}\right)=\frac{L(data,\theta,\lambda)}{\int_0^\infty \int_0^\infty L(data,\theta,\lambda)d\theta d\lambda}=\frac{\Psi}{\int_0^\infty \int_0^\infty \Psi\, d\theta d\lambda} \quad (26)$$

According to that, the posterior *pdf*'s of θ and λ are

$$\pi_{\theta,KUD}\left(\frac{\theta}{data}\right)=\frac{\int_0^\infty L(data,\theta,\lambda)\,d\lambda}{\int_0^\infty \int_0^\infty L(data,\theta,\lambda)d\theta d\lambda} \quad (27)$$

and

$$\pi_{\lambda,KUD}\left(\frac{\lambda}{data}\right)=\frac{\int_0^\infty L(data,\theta,\lambda)\,d\theta}{\int_0^\infty \int_0^\infty L(data,\theta,\lambda)d\theta d\lambda} \quad (28)$$

Therefore, the Bayes estimators for the parameters, θ and λ denoted by $\widehat{\theta}_{Bayes}$ and $\widehat{\lambda}_{Bayes}$, under squared error loss function, respectively as [31].

$$\widehat{\theta}_{Bayes}=E\left(\frac{\theta}{data}\right)=\frac{\int_0^\infty \int_0^\infty \theta\,\Psi\, d\lambda d\theta}{\int_0^\infty \int_0^\infty L(data,\theta,\lambda)d\theta d\lambda}=\frac{\int_0^\infty \int_0^\infty \theta\,\Psi\, d\lambda d\theta}{\int_0^\infty \int_0^\infty \Psi\, d\theta d\lambda} \quad (29)$$

and

$$\hat{\lambda}_{Bayes}=E\left(\frac{\lambda}{data}\right)=\frac{\int_0^\infty \int_0^\infty \lambda\,\Psi\, d\theta d\lambda}{\int_0^\infty \int_0^\infty L(data,\theta,\lambda)d\theta d\lambda} \quad (30)$$

III. Results and Discussion

In this section, we consider both the simulated and real-life data sets to compare the flexibility (KUD) to compare the estimation is dependent on maximum likelihood estimation and Bayesian estimation method based on (RSS) approach. For comparison purposes, we utilized the AICC(corrected Akaike information criterion), The MLE, and Bayes estimators ($\widehat{\theta}_{ML}$, $\widehat{\lambda}_{ML}$, $\hat{\theta}_{bayes}$ and $\hat{\lambda}_{bayes}$) which provides us lesser values of AICC and MSEs is considered best. The values of AICC can be computed as follows: AICC= AIC+$\frac{2k(k+1)}{(n-k-1)}$, where AIC=2k-2logL and k is the number of parameters, n is the sample size, -2logL is the maximized value of the likelihood function. The analysis of both data sets is performed through R software. The MLEs and Bayes of the parameters are gained with standard errors shown in parentheses. Furthermore, the corresponding values of AICC are displayed in Tables 1,2,3and 4.

A. Simulated Data

In the Monte Carlo simulation study, three data sets of size 30,60, 80, and 100 have been generated from R software and are based on 10,000 replications to obtain the MLE and Bayes estimators of the unknown parameters Kumaraswamy distribution and to compare the performance of these estimators based (RSS). The simulations are made for several combinations of the parameters r, w, and λ values while the value of the shape parameter $\theta = 2$. the estimators $\widehat{\theta}_{ML}$, $\widehat{\lambda}_{ML}$, $\hat{\theta}_{bayes}$ and $\hat{\lambda}_{bayes}$, [19]The data sets are obtained by using the inverse cdf method as discussed in section 5 and the summary of results is presented in the table 1,2,3,4 below:

TABLE I
BIASES OF THE ESTIMATORS (KUD)

n=r,w	**λ**	**Parameter Estimates** $\hat{\theta}_{ML}$	$\hat{\theta}_{bayes}$	**AICC of** $\hat{\theta}_{ML}$	**AICC of** $\hat{\theta}_{bayes}$
2,15		0.10211 (0.32198)	0.19436 (0.23475)	49.12343	47.32947
3,10	1	0.65310 (0.04387)	0.02985 (0.01198)	50.43908	52.90371
5,6		0.02387 (0.13952)	0.01211 (0.12198)	95.86310	93.90528
2,15		0.76532 (0.12865)	0.64309 (0.12196)	24.53190	23.46206
3,10	2	0.21376 (0.43183)	0.01765 (0.32198)	90.73109	89.63190
5,6		1.05297 (0.41854)	1.00211 (0.32198)	106.72031	105.82701
2,15		0.72109 (0.20018)	0.43109 (0.02100)	26.51902	25.18935
3,10	3	0.67823 (0.43218)	0.50911 (0.32198)	103.00629	101.10937
5,6		0.04819 (0.48720)	1.00211 (0.32198)	124.17409	122.11905
n=r,w	λ	Parameter Estimates $\hat{\lambda}_{ML}$	$\hat{\lambda}_{bayes}$	AICC of $\hat{\lambda}_{ML}$	AICC of $\hat{\lambda}_{bayes}$
2,15	1	0.10019 (0.31109)	0. 09841 (0.10943)	30.19201	29.18269
3,10		0.02765 (0.04198)	0.01965 (0.01100)	69.21980	64.18275
5,6		0.01634 (0.12865)	0.01022 (0.11093)	72.10295	69.18370
2,15		0.05715 (0.12432)	0.05543 (0.11085)	22.29186	20.19375
3,10	2	0.21098 (0.32765)	0.001427 (0.02098)	88.63019	85.17365
5,6		0.92854 (0.39528)	0.61098 (0.20917)	60.16295	55.83297
2,15		0.72011 (0.19802)	0.62091 (0.03475)	21.16543	19.28431
3,10	3	0.08932 (0.32198)	0.49017 (0.09285)	90.29538	83.00917
5,6		0.07890 (0.30972)	0.06827 (0.06876)	91.19200	89.22481

TABLE II
RIOR HYPER-PARAMETER ($\theta_1 = 2, \theta_2 = 2, \lambda_1 = 3, \lambda_2 = 3$).

n=r,w	λ	Parameter Estimates $\hat{\theta}_{ML}$	$\hat{\theta}_{bayes}$	AICC of $\hat{\theta}_{ML}$	AICC of $\hat{\theta}_{bayes}$
2,40		0.09919 (0.30189)	0.91028 (0.20917)	40.19828	38.1872
4,20	1	0.52109 (0.03011)	0.50928 (0.01087)	42.20198	39.29081
5,16		0.01294 (0.11946)	0.01093 (0.11409)	89.29186	86.29017
2,40		0.56092 (0.10937)	0.50927 (0.10873)	20.89276	18.98276
4,20	2	0.27092 (0.41928)	0.01309 (0.29820)	85.42907	83.20918
5,16		1.03918 (0.39019)	1.00089 (0.29017)	99.91730	95.29018
2,40		0.69027 (0.11902)	0.39027 (0.00998)	22.2092	19.09271
4,20	3	0.52987 (0.39827)	0.00954 (0.29610)	97.90827	94.00927
5,16		0.02897 (0.39045)	0.00189 (0.29075)	107.09276	104.92810
n=r,w	λ	Parameter Estimates $\hat{\theta}_{ML}$	$\hat{\theta}_{bayes}$	AICC of $\hat{\theta}_{ML}$	AICC of $\hat{\theta}_{bayes}$

2,40	1	0.08918 (0.29081)	0. 07298 (0.19876)	26.16490	25.19045
4,20		0.01954 (0.02986)	0.00975 (0.01943)	63.23876	58.23890
5,16		0.01197 (0.11908)	0.00834 (0.09343)	67.09268	62.39047
2,40	2	0.03198 (0.10934)	0.03098 (0.09175)	18.20543	16.82901
4,20		0.29816 (0.29165)	0.00100 (0.01628)	81.10286	78.01927
5,16		0.60296 (0.20185)	0.40197 (0.17290)	53.20195	48.72017
2,40	3	0.50187 (0.18520)	0.40918 (0.02718)	17.20175	14.00186
4,20		0.06197 (0.29017)	0.03196 (0.06194)	83.63245	78.23549
5,16		0.00619 (0.20186)	0.00519 (0.02891)	84.10956	82.32061

TABLE III
$\theta = 2$ AND THE PRIOR HYPER-PARAMETER ($\theta_1 = 2, \theta_2 = 2, \lambda_1 = 3, \lambda_2 = 3$)

=r,w	λ	Parameter Estimates $\hat{\theta}_{ML}$	$\hat{\theta}_{bayes}$	AICC of $\hat{\theta}_{ML}$	AICC of $\hat{\theta}_{bayes}$
2,50	1	0.08976 (0.29018)	0.87654 (0.19208)	37.21751	34.12098
4,25		0.51827 (0.02964)	0.41902 (0.00918)	39.19287	33.10276
5,20		0.01197 (0.10876)	0.00976 (0.08276)	78.19365	76.18291
2,50	2	0.49187 (0.01875)	0.41762 (0.00187)	18.19375	16.22197
4,25		0.25187 (0.39201)	0.00186 (0.22018)	79.10271	75.54322
5,20		1.00187 (0.32019)	1.19871 (0.28196)	93.10385	89.83104
2,50	3	0.66109 (0.01876)	0.35104 (0.00091)	19.10438	15.10275
4,25		0.49012 (0.31037)	0.00198 (0.26194)	89.83190	83.10976
5,20		0.00187 (0.32901)	0.00017 (0.25101)	103.10934	101.83100
n=r,w	λ	Parameter Estimates $\hat{\lambda}_{ML}$	$\hat{\lambda}_{bayes}$	AICC of $\hat{\lambda}_{ML}$	AICC of $\hat{\lambda}_{bayes}$
2,50	1	0.02817 (0.22910)	0. 028171 (0.16028)	21.20176	18.29186
4,25		0.01864 (0.01854)	0.00401 (0.01001)	57.19365	53.21097
5,20		0.00185 (0.10001)	0.00107 (0.08176)	63.00934	58.54309
2,50	2	0.02187 (0.09185)	0.01087 (0.08115)	16.19435	14.10428
4,25		0.27185 (0.27108)	0.00089 (0.01098)	77.01832	74.20165
5,20		0.54309 (0.18639)	0.37623 (0.03643)	49.10376	46.10387
2,50	3	0.40917 (0.16001)	0.29175 (0.01864)	14.19483	11.04296
4,25		0.05187 (0.19275)	0.02176 (0.02837)	79.03196	73.11098
5,20		0.00276 (0.10934)	0.00065 (0.01936)	80.11053	76.20175

From Table 1 to 4, it can be concluded that the MSEs and the values of AICC of the estimates of θ and λ made by both methods decrease as set sizes increase. It is also noted that biases and MSEs and the values of AICC of the shape parameter λ decrease when its population value increases. Also, almost in all cases, the biases and the MSEs and the criteria like AICC for the Bayes estimates of both parameters θ and λ are lesser values than the MLE estimates θ , λ respectively.

B. Real-Life Data

Here consider the two real data sets pertaining as given under, and the results are presented in table 5.

Data set I: The first real data consists of the number of successive failures for the air conditioning system reported for each member in a fleet of 13 Boeing 720 jet airplanes. The pooled data with 214 observations by Proschan (1963), [15]and others. The data are: 50, 130, 87, 57, 102, 15, 14, 10, 57,320, 261, 51, 44, 9, 254, 493, 33, 18, 209, 41, 58, 60, 48, 56, 87, 11, 102, 12, 5, 14, 14, 29, 37, 186,29, 104, 7, 4, 72, 270, 283, 7, 61, 100, 61, 502, 220, 120, 141, 22, 603, 35, 98, 54, 100, 11, 181, 65,49, 12, 239, 14, 18, 39, 3, 12, 5, 32, 9, 438, 43, 134, 184, 20, 386, 182, 71, 80, 188, 230, 152, 5, 36,79, 59, 33, 246, 1, 79, 3, 27, 201, 84, 27, 156, 21, 16, 88, 130, 14, 118, 44, 15, 42, 106, 46, 230, 26,59, 153, 104,20,206, 5, 66, 34, 29, 26, 35, 5, 82, 31, 118, 326, 12, 54, 36, 34, 18, 25, 120, 1,22,18, 216, 139, 67, 310, 3, 46, 210, 57, 76, 14, 111, 97, 62, 39, 30, 7, 44, 11, 63, 23, 22, 23, 14, 18,13, 34, 16, 18, 130, 90, 163, 208, 1, 24, 70, 16, 101, 52, 208, 95, 62, 11,

Data set II: the second real data set reported [16], the data of failure times (in a year) to 45 Patients consisting of: 2.178,0.395,4.003,2.652,0.121,0.540,0.604,0.507,0.841,0.296,3.978,0.501,0.456,0.125,0.047,0.164,0.197,0.203,0.260,0.334,0.458,0.538,0.544,0.282,0.132,0.969,0.863,2.444,1.099,1.447,1.553,1.581,2.343,2.416,2.830,3.578,1.219,4.223,1.326,3.743.

TABLE IV
BIASES OF THE ESTIMATORS KUMARASWAMY DISTRIBUTION FOR POPULATION $\theta = 2$

N	λ	Parameter Estimates $\hat{\theta}_{ML}$	$\hat{\theta}_{bayes}$	AICC of $\hat{\theta}_{ML}$	AICC of $\hat{\theta}_{bayes}$
Data Set I	1	0.34271 (0.5312)	0.21987 (0.39186)	50.28392	45.22198
	2	1.19876 (0.5017)	1.92718 (0.40199)	69.19527	55.18390
	3	0.18265 (0.5613)	1.28177 (0.51926)	80.22157	76.92001
Data Set II	1	0.11836 (0.6211)	0.10287 (0.41254)	55.32908	49.29987
	2	1.65194 (0.5681)	1.17299 (0.42218)	83.19837	74.27365
	3	0.19992 (0.7218)	1.00176 (0.48136)	87.34798	83.01876
	λ	Parameter Estimates $\hat{\lambda}_{ML}$	$\hat{\lambda}_{bayes}$	AICC of $\hat{\lambda}_{ML}$	AICC of $\hat{\lambda}_{bayes}$
Data Set I	1	0.19919 (0.49201)	0. 39761 (0.38102)	22.52952	20.10953
	2	0.42001 (0.50000)	0.41098 (0.40187)	55.53198	49.18305
	3	0.10638 (0.46240)	0.10453 (0.43209)	78.19428	75.19402
Data Set II	1	0.32187 (0.60094)	0. 4086 (0.52074)	21.85103	20.10386
	2	0.52071 (0.51099)	0.73104 (0.30128)	32.20437	29.10629
	3	0.13840 (0.59274)	0.35405 (0.41587)	75.31952	69.10394

The authors can conclude from these results that the MSEs and the values of AICC of the estimates of θ and λ made by both methods decrease as set sizes increase. It is also noted that biases and MSEs and the values of AICC of the shape parameter λ decrease when its population value increases. Also, almost in all cases, the biases and the MSEs and the criteria like AICC for the Bayes estimates of both θ and λ are lesser values than the MLE estimates of θ and λ respectively.

IV. Conclusion

This research considered the estimation problem of unknown parameters (KUD) depending on (RSS). MLE and Bayesian estimation methods are used where Bayes estimates were obtained under the squared error loss function. Based on applied to both the generated and the real-life data sets, it is observed that the Bayes estimators perform better than MLE estimators relative to their biased MSE and values of AICC. Furthermore, biases and MSEs and the values of AICC of estimates for the parameter λ., under the RSS approach, are lesser than the corresponding estimates for the parameter θ.

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