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Advancing Risk Management with GAS-1F: Value at Risk and Expected Shortfall Estimation

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***Abstract*—Value at Risk (VaR) and Expected Shortfall (ES) are critical metrics for quantifying financial risk. VaR estimates the maximum potential loss within a specific timeframe, while ES captures the average loss that exceeds the VaR threshold. Accurate estimation of these risk measures is vital for financial institutions; however, traditional methods often falter in addressing the dynamic volatility of financial data. This study explores the One-Factor Generalized Autoregressive Score (GAS-1F) semiparametric model, a novel approach that incorporates elicitability into its score function to circumvent distributional assumptions. Elicitability guarantees alignment between the estimated loss function and the true underlying risk measure. The GAS-1F model excels as a two-tiered, semiparametric framework for estimating VaR and ES. By applying this model to historical data from the S&P 500 index, we demonstrate its effectiveness in estimating these risk metrics. The model operates in two stages: first, it estimates the volatility of the data, reflecting the extent of price fluctuations. This estimated volatility is then utilized to calculate VaR and ES for each data point, generating a time series of daily values that offer a comprehensive view of potential risks investors face. The Diebold-Mariano test reveals that the GAS-1F model achieves superior accuracy compared to the widely used GARCH parametric model in estimating VaR and ES for stock prices. This enhanced accuracy can significantly benefit financial institutions, providing informed risk management decisions and valuable insights for short-term investors, particularly day traders, by facilitating more effective risk management strategies.**

***Keywords*—Elicitability; expected shortfall; generalized autoregressive score; risk; value at risk; volatility.**

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I. INTRODUCTION

Risk analysis is a critical tool for investors in providing a comprehensive understanding of potential risks [1]. It empowers them to implement appropriate strategies to mitigate or effectively manage the risks faced. Value at Risk (VaR) and Expected Shortfall (ES) are the two primary metrics employed [2]. VaR emerged as a prominent measure to quantify potential losses within a specific timeframe for a given asset [3]. Initially introduced by JP Morgan in 1996, VaR gained widespread adoption as a standard risk measurement tool after being implemented by the Basel Committee on Banking Supervision (BCBS). Mathematically, if Y represents the collection of return rates y_t observed for an asset over a specific period, with a conditional distribution denoted by F_Y, then the α-level VaR can be defined as:

$$VaR \equiv \inf\{Y|F_Y(y_t) \geq \alpha\}, t = 1, \dots, T \quad (1)$$

However, VaR has been criticized for failing to satisfy one of the four axioms of coherence proposed by Artzner et al. [4]. These axioms establish a set of desirable properties for a risk measure. A measure is considered coherent if it adheres to all four axioms. VaR specifically violates the subadditivity axiom, which states that the combined risk of multiple investment portfolios should not exceed the sum of the individual portfolio risks. Additionally, VaR focuses solely on the potential loss threshold, neglecting the severity of losses exceeding that threshold [5].

The limitations of VaR, particularly its inability to capture extreme risks and significant potential losses, became increasingly evident during the 2007-2008 financial crisis. This shortcoming prompted the Basel Committee on Banking Supervision (BCBS) to advocate for a transition from VaR to Expected Shortfall (ES) within its Basel III framework [6]. Rockafellar and Uryasev [7] refer to ES as Conditional Value at Risk (CVaR), defining it as the anticipated loss for an asset, considering only returns that fall below the distribution quantile, which is VaR, as follows:

$$ES \equiv E[Y|y_t \leq VaR], t = 1, \ldots, T \quad (2)$$

In contrast to VaR, Expected Shortfall (ES) incorporates the severity of potential losses beyond the VaR threshold, considering the shape of the loss distribution's tail [8]. This characteristic makes ES more adept at capturing the risk associated with extreme events, even those with a low probability of occurrence, which can significantly impact portfolio performance [9]. Furthermore, ES satisfies all four axioms of coherence, solidifying its suitability as a risk measure [10].

In practice, the VaR and ES formula can be used to estimate these risk measures based on historical data without relying on a particular model. However, this method has limitations in the assumption of volatility. Volatility is the fluctuation of a set of assets which measures the level of dispersion [11]. The VaR formula assumes constant volatility (homoscedasticity), whereas in real financial data, this is often violated because volatility often changes over time, i.e. heteroscedasticity occurs [12]. Therefore, the VaR and ES formulas may not capture extreme events or changes in market conditions that are not adequately represented in historical data [13].

In response to these challenges, diverse methodologies have emerged for estimating the risk metrics. These methodologies encompass volatility modeling, a process involving the prediction of future volatility by analyzing historical return data through a conditional variance or conditional heteroscedasticity models capable of accommodating time-varying variability [14]. Subsequently, the outcomes of these models are applied within the VaR framework to derive ES, constituting a two-stage estimation approach. Noteworthy among these models are various forms of conditional heteroscedasticity models, such as the Generalized Autoregressive Conditional Heteroskedasticity (GARCH) model by Bollerslev [15], which led to the development of the class of Generalized Autoregressive Score (GAS) model [16], [17] .

According to Creal et al. [18], the GAS model represents a broader version of traditional autoregressive (AR) models, where parameters are estimated through a score function [17]. Specifically, by selecting the appropriate score function, the GAS model can encompass other conditional heteroscedasticity models, including the GARCH model. Creal et al. [18] introduced this class of model to offer a straightforward, unified approach for accommodating the various models that have developed. A key advantage is its simplicity in evaluating the likelihood function and its potential for extension to other models without introducing additional complexity [19].

On the other hand, the concept of elicitability gained attention, prompting several researchers to explore its application in risk estimation [20]. A risk measure is said to be elicitable if there exists a loss function that minimizes expected losses, making the measure a solution [21]. The introduction of Basel III by the BCBS triggered discussions, particularly regarding ES non-elicitability, contrasted with VaR established elicitability. However, Fissler and Ziegel [22] demonstrated that ES, in conjunction with VaR, is jointly elicitable. This suggests that VaR and ES can be estimated simultaneously by minimizing a loss function. They proposed a class of loss functions named FZ Loss, which has been shown to derive VaR and ES when minimized effectively [23].

Amidst the diverse methodologies for estimating VaR and ES, Song and Li [13] categorized the methods into three groups: parametric, nonparametric, and semiparametric. Parametric models operate under the assumption that return value distributions adhere to specific distributions. However, this assumption may not always hold, potentially leading to inaccuracies in model outputs. Nonparametric models, in contrast, do not use distributional assumptions, offering greater flexibility. Nevertheless, they necessitate careful parameter selection, adding complexity to the estimation process [24], [25].

The Generalized Autoregressive Score (GAS) was introduced by Patton et al. [26] as semiparametric approach for estimating VaR and ES by minimizing the FZ Loss function [23]. These models employ a parametric structure to capture the dynamics of VaR and ES through lagged information, without presuming the distribution of returns. This approach offers a middle ground between parametric and nonparametric models. However, if the data adheres to a specific distribution, employing a parametric model with maximum likelihood estimation and appropriate distribution assumptions may yield more precise estimates [27]. This raises the research inquiry of this study: whether semiparametric models can deliver more accurate estimates compared to parametric models, which rely on specific distributional assumptions. Consequently, this study concentrates on the GAS-1F semiparametric model introduced by Patton et al. [26] for joint VaR and ES estimation utilizing elicitability principles.

This paper is structured as follows. Section 2 introduces the GAS-1F model for VaR and ES estimation. Section 3 features simulation studies employing the Nelder-Mead method to estimate the proposed model. Subsequently, the constructed model is applied to out-of-sample data to further estimate VaR and ES. The Diebold-Mariano test is then employed to evaluate the performance of the GAS-1F model's estimates. Finally, Section 4 presents a discussion of the findings and draws conclusions.

II. Materials and Method

In this section we study the GAS-1F model proposed by Patton et al. [26] for ES and VaR estimation. The following GAS-1F model were proposed as a semiparametric model by exploiting the loss function proposed Fissler and Ziegel [22] and the GAS framework [18].

A. L_{FZ0} Loss Function

Fissler and Ziegel [22] introduced a class of loss functions called FZ Loss, denoted by L_{FZ} as follows:

$$\begin{aligned} &L_{FZ}(Y, v, e; \alpha, G_1, G_2) \\ &= (\mathbf{1}\{Y \leq v\} - \alpha)\left(G_1(v) - G_1(Y) + \frac{1}{\alpha} G_2(e) v\right) \\ &-G_2(e)\left(\frac{1}{\alpha}\mathbf{1}\{Y \leq v\}Y - e\right) - \mathcal{G}_2(e) \end{aligned} \quad (3)$$

Y represents the collection of rates of return, v denotes the VaR, and e signifies the ES. An indicator function, denoted by $\mathbf{1}$, plays a key role. This function outputs a value of one if the rate of return $y \in Y$ is less than or equal to the VaR threshold v, indicating an exceedance of the VaR level. In all other cases, the indicator function returns a value of zero. G_1 and G_2 are both unspecified index function with $\mathcal{G}_2' = G_2$. It is shown that any loss function from the FZ loss class meets the criteria as a consistent score function for the VaR and ES pair, i.e. minimizing the expected loss using any function within this class yields the true VaR and ES values [21].

A crucial aspect is choosing G_1 and G_2 such that the resulting loss function generates a difference that is homogeneous of degree zero. This property has been shown in volatility forecasting applications to lead to higher power in Diebold and Mariano tests [28]. Nolde and Ziegel [29] shows that, in general, there is no FZ loss function that produces a homogeneous loss difference of zero degree. However, in risk management applications, for values of α that are of interest (ranges from 0.01 to 0.1), it can be assumed that $v, e < 0$ [26]. Thus, following this fact, the FZ loss function class yields exactly one function that produces a homogeneous loss difference of zero degree, denoted by L_{FZ0}:

$$L_{FZ0}(Y, v, e; \alpha) = -\frac{1}{\alpha e}\mathbf{1}\{Y \le v\}(v - Y) + \frac{v}{e} + log(-e) - 1 \tag{4}$$

which is obtained by specifying $G_1(x) = 0$ and $G_2(x) = -\frac{1}{x}$. The FZ0 loss function enables the exploration of the semiparametric dynamic model GAS-1F for estimating Expected Shortfall (ES) and Value at Risk (VaR).

B. GAS-1F Model for VaR and ES

Consider a dataset containing historical return rates y_t of an asset for a defined period $t = 1, \dots, T$. To effectively estimate VaR and ES while incorporating time-varying volatility, the semiparametric GAS-1F model estimates the variable k_t which represents the component of volatility at time point t. VaR and ES is then calculated using the estimated volatility, as follows:

$$k_t = \omega + \beta k_{t-1} + \gamma I_{t-1} s_{t-1} \tag{3}$$

$$v_t = a \exp\{k_t\} \tag{4}$$

$$e_t = b \exp\{k_t\}, b < a < 0 \tag{5}$$

$\exp\{k_t\}$: volatility at time t
I_{t-1} : scale factor at time $t-1$
s_{t-1} : score function at time $t-1$
v_t : VaR at time t
e_t : ES at time t
$a, b, \omega, \beta, \gamma$: parameters to be estimated

The GAS-1F model shown above is constructed based on the GAS framework by Bogdan et al. [17]. However, the GAS-1F model utilizes the loss function L_{FZ0} instead of a likelihood function in constructing the scale factor and score function, circumventing the need for specific assumptions about the underlying data distribution [26].

Based on how the VaR and ES are defined in (6) and (7), it can be obtained that:

$$\frac{\partial v_t}{\partial k_t} = \frac{\partial^2 v_t}{\partial k_t^2} = a \exp\{k_t\} = v_t \tag{6}$$

$$\frac{\partial e_t}{\partial k_t} = \frac{\partial^2 e_t}{\partial k_t^2} = b \exp\{k_t\} = e_t \tag{7}$$

Therefore, replacing the log-likelihood function with the L_{FZ0} function yields the following score function and scaling factor:

$$s_t \equiv \frac{\partial L_{FZ0}(Y_t, v_t, e_t; \alpha)}{\partial k_t} = -\left(\frac{1}{\alpha(b \exp\{k_t\})} 1\{Y_t \le a \exp\{k_t\}\}Y_t - 1\right) \tag{8}$$

$$I_t \equiv \frac{\partial^2 \mathbb{E}_{t-1}[L_{FZ0}(Y_t, v_t, e_t; \alpha)]}{\partial k_t^2} = -\frac{1}{\alpha e_t}\big(f_Y(v_t)\big)(v_t)^2 + 1 \tag{9}$$

However, it is found that the scaling factor I_t above depends on the unknown distribution of Y_t. To avoid using distribution assumptions, Patton et al. [26] used an approximation that Y_t can be written as a multiplication of a random variable η_t with a volatility ϕ_t, that is, $Y_t = \phi_t . \eta_t \Leftrightarrow \eta_t = \frac{Y_t}{\phi_t}$ where $\eta_t \sim iid$. This way, it can be obtained that $f_Y(v_t)$ is proportional to v_t^{-1}, that is:

$$f_Y(v_t) = \frac{a}{v_t} f_\eta(a) = \frac{n}{v_t} \tag{10}$$

by assuming $n = a f_\eta(a) < 0$. Consequently, putting $f_Y(v_t) = \frac{n}{v_t}$ in the scaling factor I_t results as follows:

$$I_t = 1 - \frac{n}{\alpha}\frac{a}{b} \tag{11}$$

where $n < 0 < \frac{a}{b} < 1$. In this context, I_t turns out to be a positive constant of unknown value. This implies that the model needs to estimate a positive constant as its scaling factor. Patton et al. [26] have observed that employing a larger scaling factor tends to lead to significant loss values. Thus, for simplicity, I_t is set to 1. Consequently, the final form of the GAS-1F model can be expressed as follows:

$$k_t = \omega + \beta k_{t-1} + \gamma \frac{1}{b \exp\{k_{t-1}\}}\left(\frac{1}{\alpha}\mathbf{1}\{Y_{t-1} \le a \exp\{k_{t-1}\}\}Y_{t-1} - b \exp\{k_{t-1}\}\right) \tag{14}$$

C. GAS-1F Parameter Estimation for VaR and ES

Based on the previous section, GAS-1F model involves five parameters: a, b, ω, β, and γ, where ω represents a constant. Patton et al. [26] found that there exist multiple sets of parameters that can yield the same time series k_t (i.e. no unique solution). Assigning a value to ω can address this issue [26]. For simplicity, we set $\omega = 0$. Consequently, there are four parameters which require estimation in this study, namely $\theta = \{a, b, \beta, \gamma\}$.

Building upon the work of Lazar and Xue [21], who demonstrated the effectiveness of minimizing expected loss using the L_{FZ0} function for estimating VaR and ES, this study seeks to identify the optimal parameter values by using this concept. Specifically, the objective is to find the parameter vector θ that minimizes the L_{FZ0} function at each time point t. This optimization problem can be mathematically formulated as follows:

$$\hat{\theta} \equiv \arg\min_{\theta \in \mathbb{R}}\big(g(\theta)\big) \tag{12}$$

$$g(\theta) = \frac{1}{T}\left[\sum_{t=1}^{T} L_{FZ0_t}(Y_t, v_t(\theta), e_t(\theta); \alpha)\right], t = 1, \ldots, T \quad (13)$$

$\widehat{\boldsymbol{\theta}}$: GAS-1F parameter estimate
$g(\boldsymbol{\theta})$: objective function
$v_t(\boldsymbol{\theta})$: VaR evaluated using parameter $\boldsymbol{\theta}$
$e_t(\boldsymbol{\theta})$: ES evaluated using parameter $\boldsymbol{\theta}$

However, while the chosen loss function offers advantages, it presents challenges for optimization. The function incorporates the indicator function $\mathbf{1}\{Y_t \leq v_t\}$ which exhibits discontinuity at points where $Y_t = v_t$.

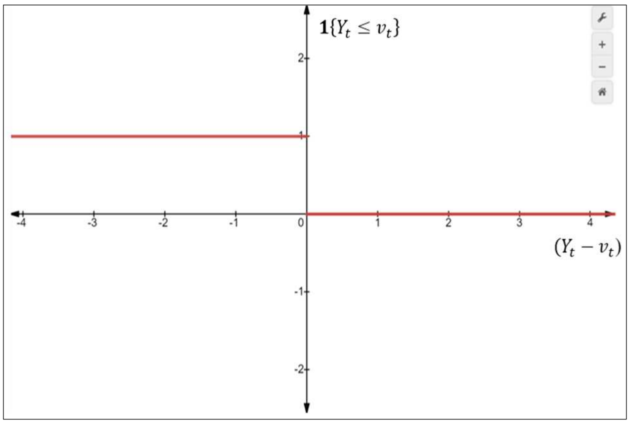

Fig. 1 Visualization of indicator function $\mathbf{1}\{Y_t \leq v_t\}$ discontinuity

This discontinuity renders gradient-based optimization methods infeasible. On the other hand, direct search methods, while potentially viable, are unsuitable due to their sensitivity to initial parameter values [30]. Building upon [26] this study incorporates a logistic linking function to address these limitations, which is defined as follows:

$$\Gamma(Y_t, v_t; \tau) = \frac{1}{1+\exp\{\tau(Y_t - v_t)\}}, \tau > 0 \quad (14)$$

The smoothing parameter, denoted by τ, represents the degree of smoothness in the model. The function above converges to the indicator function as τ approaches ∞. Therefore, by replacing the indicator function in the loss function L_{FZ0} with the logistic linking function, we obtain the smoothed version of the loss function, defined as follows:

$$\begin{aligned} &{}_{S}L_{FZ0_t}(Y_t, v_t, e_t; \alpha, \tau) \\ &= -\frac{1}{\alpha e_t}\left(\frac{1}{1+\exp\{\tau(Y_t - v_t)\}}\right)(v_t - Y_t) + \frac{v_t}{e_t} \\ &\quad + \log(-e_t) - 1 \end{aligned} \quad (18)$$

With the smoothed loss function ${}_{S}L_{FZ0_t}$ in hand, gradient search optimization becomes feasible. This allows us to obtain parameter values $\widehat{\boldsymbol{\theta}}_\tau$ that intuitively represent close approximations to the true parameter estimates $\widehat{\boldsymbol{\theta}}$. Next, the Nelder-Mead optimization is utilized as a direct search method to estimate the true values of $\boldsymbol{\theta}$ using the unsmoothed loss function $\boldsymbol{L_{FZ0_t}}$ and the initial parameter values $\widehat{\boldsymbol{\theta}}_\tau$ obtained from gradient search. This approach effectively addresses the sensitivity of direct search methods to initial values [26]. The true parameter estimates, denoted by $\widehat{\boldsymbol{\theta}}$, are then employed to construct the GAS-1F model. With the constructed GAS-1F model, the VaR and ES estimates can be obtained, which is $\hat{v}_t$ and $\hat{e}_t$ for $t = 1, 2, \ldots, T$. These values represent the estimated VaR and ES at each time point, accounting for the volatility at that specific time point.

III. Results and Discussion

In this section, we apply the previously discussed GAS-1F model to a dataset comprising daily rate of return data from S&P 500 index share prices, covering the period from January 3, 1990, to December 30, 2016, with a total of 6,800 observations.

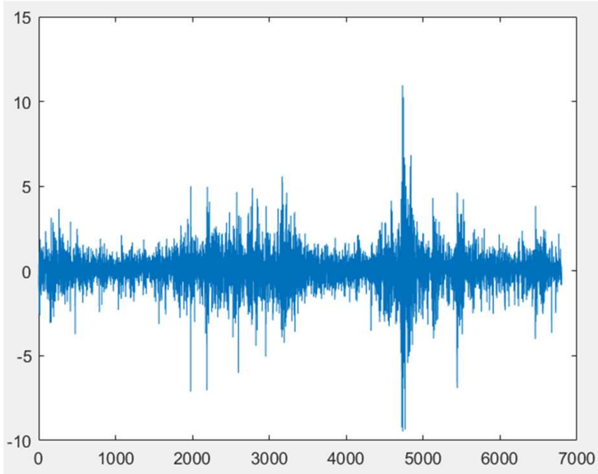

Fig. 2 Visualization of S&P 500 index share prices dataset from January 3, 1990, to December 30, 2016

TABLE I
Summary Statistics

Mean	0.02690
Std. deviation	1.12620
Max	10.9572
Min	-9.4695
$\hat{v}_{\alpha=0.05}$	-1.7315
$\hat{e}_{\alpha=0.05}$	-2.6972

From the statistics shown above, it can be interpreted that during the period from January 3, 1990, to December 30, 2016, the risk of loss at a 95% confidence level is -1.7315%, and in cases where the risk exceeds this value, the loss faced is -2.6972%. However, it is evident that the minimum observed value is at -9.4695%. In other words, there exists loss more severe than the estimates by more than 6% of the value of $\hat{e}_{\alpha=0.05}$. This indicates that neither $\hat{v}_{\alpha=0.05}$ nor $\hat{e}_{\alpha=0.05}$ provide sufficient information to account for the volatility in the dataset. Therefore, the GAS-1F model is constructed to estimate VaR and ES.

Before estimating the parameters of the GAS-1F model, the S&P 500 rate of return dataset is divided into two subsets. The rate of return values observed during the first ten years (January 3, 1990, to December 31, 1999) is used as training data, comprising 2525 observations. The rate of return values observed after December 31, 1999, is used as testing data, comprising 4,275 observations.

A. In-sample Estimation

We now present estimates of the parameters of the models presented in Section 2. To analyze the impact of different τ values on the resulting loss, the gradient search optimization is performed three times, with smoothing parameters set to $\tau = 5$, $\tau = 20$, and $\tau = 100$.

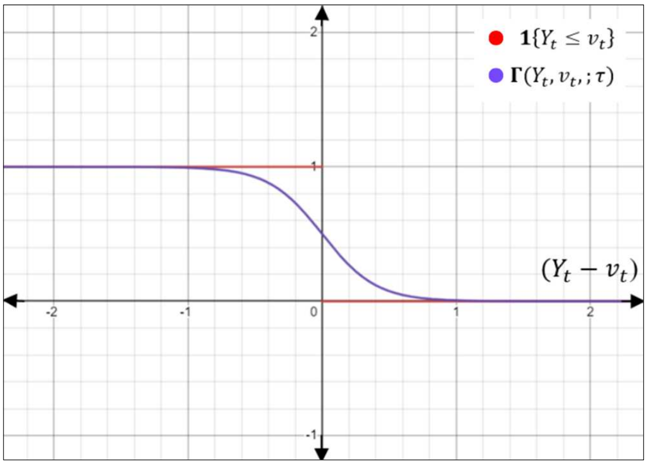

Fig. 3 Visualization of the logistic function with $\tau = 5$ compared to the indicator function $\mathbf{1}\{Y_t \leq v_t\}$

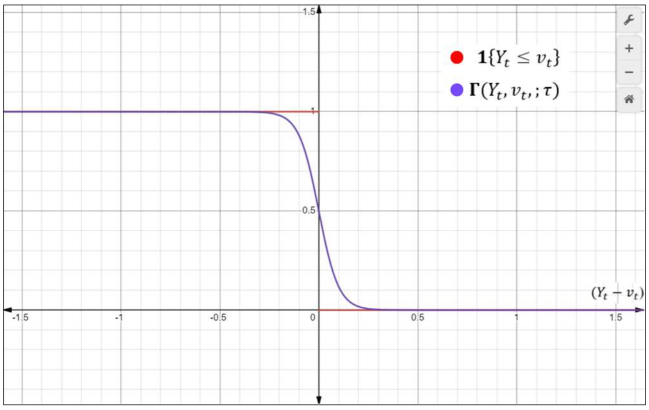

Fig. 4 Visualization of the logistic function with $\tau = 20$ compared to the indicator function $\mathbf{1}\{Y_t \leq v_t\}$

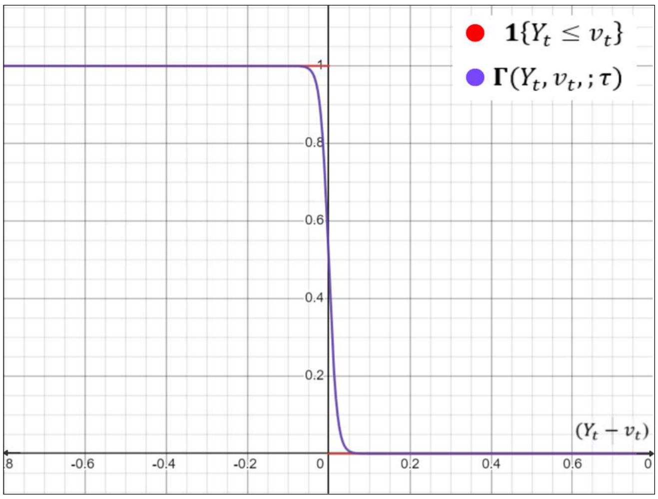

Fig. 5 Visualization of the logistic function with $\tau = 100$ compared to the indicator function $\mathbf{1}\{Y_t \leq v_t\}$

TABLE II

PARAMETER ESTIMATES OBTAINED FROM GRADIENT SEARCH OPTIMIZATION

$\widehat{\boldsymbol{\theta}}_\tau$	$\tau = 5$	$\tau = 20$	$\tau = 100$
$\hat{a}_\tau$	-0.6218	-1.1553	-1.1904
$\hat{b}_\tau$	-0.8959	-1.7437	-1.8172
$\hat{\gamma}_\tau$	0.0142	0.0073	0.0069
$\hat{\beta}_\tau$	0.9862	0.9925	0.9945

To determine the best parameter set, the results from these optimizations are then used for direct search optimization using the actual objective function. In other words, the direct search optimization is conducted three times, each using $\widehat{\boldsymbol{\theta}}_{\tau=5}, \widehat{\boldsymbol{\theta}}_{\tau=20}$, and $\widehat{\boldsymbol{\theta}}_{\tau=100}$ as the initial values. The method utilized for this direct search optimization is the Nelder-Mead method [31].

TABLE III

PARAMETER ESTIMATES OBTAINED FROM DIRECT SEARCH OPTIMIZATION

$\widehat{\boldsymbol{\theta}}_\tau$	$\widehat{\boldsymbol{\theta}}_{\tau=5}$	$\widehat{\boldsymbol{\theta}}_{\tau=20}$	$\widehat{\boldsymbol{\theta}}_{\tau=100}$
$\hat{a}$	-0.6294	-1.1642	-1.1876
$\hat{b}$	-0.9168	-1.7565	-1.8063
$\hat{\gamma}$	0.0102	0.0068	0.0069
$\hat{\beta}$	0.9983	0.9946	0.9952
$g(\widehat{\boldsymbol{\theta}}_\tau)$	0.6164	0.6025	0.6040

It can be observed from the table above that the smallest objective function value was obtained from the second optimization, which used $\widehat{\boldsymbol{\theta}}_{\tau=20}$ as the initialization point. Consequently, the GAS-1F model obtained is as follows:

$$\hat{k}_t = 0.9946\hat{k}_{t-1} + 0.0068\frac{1}{\hat{e}_{t-1}}\left(\frac{1}{\alpha}1\{Y_{t-1} \leq \hat{v}_{t-1}\}Y_{t-1} - \hat{e}_{t-1}\right) \quad (19)$$

$$\hat{v}_t = -1.1642\exp\{\hat{k}_t\} \quad (15)$$

$$\hat{e}_t = -1.7565\exp\{\hat{k}_t\} \quad (16)$$

B. Out-of-sample Estimation

It can be assumed that $\hat{k}_1 = 0$, i.e. there is no volatility when there is only one observation. Consequently, the values $\hat{v}_t$ and $\hat{e}_t$ for $t = 1,..,6800$ can be calculated using the constructed GAS-1F model as presented in the previous section.

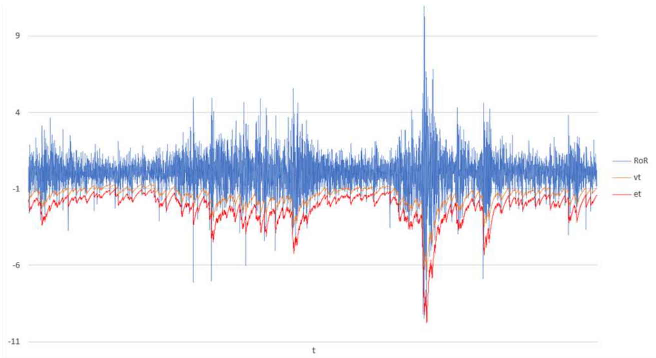

Fig. 6 VaR and ES estimates using GAS-1F model over S&P 500 index share prices from January 3, 1990, to December 30, 2016

As illustrated in Figure 6, the time series estimates of VaR and ES intuitively capture the overall pattern of daily return values. This allows us to predict at which points in time the returns are increasing or decreasing and estimate the potential magnitude of losses on those days. However, despite the time series of VaR and ES capturing the return pattern, Figure 1 shows that there are still instances where the return values are lower than the estimated VaR and ES for that day. This can occur when the return values drop drastically, indicating significant volatility spikes [31], [32]. Such sudden sharp increases may result in less accurate VaR and ES estimates using the GAS-1F model on those days. Nevertheless, instances of losses being more severe than the VaR and ES estimates are relatively rare. To evaluate the GAS-1F model's effectiveness in estimating VaR and ES, we conducted a performance comparison using the Diebold-Mariano Test. As the benchmark model, we employed the widely used GARCH(1,1) model using standard normal distribution assumption, a common parametric approach for stock price modeling [33], denoted as GARCH(1,1)-N. Firstly, the GARCH(1,1)-N model is constructed using the same training data, that is, the rate of return data for $t = 1, \ldots, 2.525$. The model obtained is as follows:

$$\sigma_t^2 = 0.0055 + 0.0518\varepsilon_{t-1}^2 + 0.9420\sigma_{t-1}^2 \tag{17}$$

$$\varepsilon_t = Y_t - 0.0588 \tag{18}$$

The VaR and ES estimates are calculated as follows:

$$_G\hat{v}_t = F^{-1}(0.05)\sigma_t + 0.0588 \tag{19}$$

$$_G\hat{e}_t = -\frac{f\left(F^{-1}(0.05)\right)}{0.05}\sigma_t + 0.0588 \tag{20}$$

This model is then used to estimate VaR and ES quantities over in-sample and out-of-sample period, similarly to the GAS-1F model. The value of σ_1^2 is set to be the sample variance.

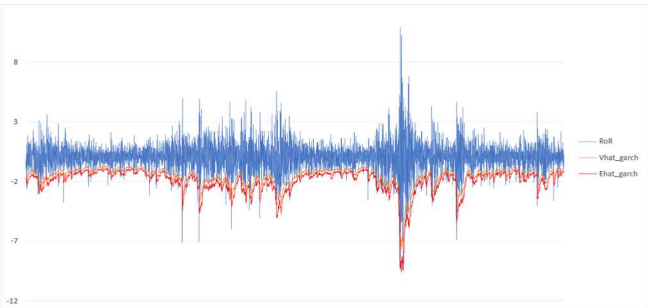

Fig. 7 VaR and ES estimates using GARCH (1,1)-N model over S&P 500 index share prices from January 3, 1990, to December 30, 2016

The performance of the two models is then compared by examining the loss values produced by each model over the out-of-sample period. Using the L_{FZ0} function, the average loss generated by both models is calculated as follows:

$$\bar{l}_{OS} = 0.8528,\ {}_G\bar{l}_{OS} = 0.8763 \tag{21}$$

It is found that the average loss for the GAS-1F model is less than that for the GARCH (1,1)-N model. However, to assert that the VaR and ES estimates using the GAS-1F model are more accurate, the significance of the difference in loss values between the two models needs to be tested. The Diebold-Mariano test is employed for this case.

The loss difference is firstly calculated at each time point over the out-of-sample period, that is, $d_t = l_t - {}_Gl_t$ for $t = 2{,}526, \ldots, 6{,}800$. Then, the mean of sample d_t can be calculated as follows:

$$\bar{d} = \left[\frac{1}{(6.800-2.526)}\sum_{t=2.526}^{6.800} d_t\right] = -0.0235 \tag{22}$$

In calculating the sample variance, Newey-West standard errors are utilized to account for potential heteroschedasticity in the data [33]. The resulting test statistic is as follows:

$$DM = \frac{\bar{d}}{\sqrt{\hat{\sigma}_{\bar{d}}^2}} = \frac{-0.0235}{\sqrt{0.0001}} \approx -2.311 \tag{23}$$

Since $DM = -2.311 < -z_{\frac{0.05}{2}} = -1.96$, we can conclude that the null hypothesis is rejected. This indicates that there is a significant difference in the accuracy of the forecasts produced by the two models. Therefore, it can be concluded that, as a semiparametric model, the GAS-1F model successfully provides a better risk assessment for the S&P 500 index return data through more accurate VaR and ES estimates compared to the parametric GARCH (1,1)-N model.

IV. Conclusion

The GAS-1F model shines as a two-stage, semiparametric tool for estimating VaR and ES. It avoids data distribution assumptions by using a nonparametric structure to estimate volatility via a loss function, replacing the usual log-likelihood function. This estimated volatility is then used to calculate VaR and ES. We validated the GAS-1F model on S&P 500 data, generating daily VaR and ES series. Compared to a benchmark model, the GAS-1F model displayed superior accuracy, as confirmed by the DM test. This flexible approach is ideal for VaR and ES estimation, especially when data distributions are uncertain. Future research could delve into the FZ Loss function class, particularly FZ0 Loss, and explore alternative factors and coefficients within the GAS-1F model. Testing its performance on diverse data sets would further solidify its capabilities.

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